A RISK NEUTRAL PRICING FRAMEWORK FOR MORTGAGE BACKED SECURITIES

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Abstract. We have developed a risk neutral pricing model for mortgage backed securities which incorporates the valuation impact of both interest rate and prepayment risk as well as a new mortgage basis risk variable. We find that this new mortgage specific variable is necessary to account for market prices. The model is parsimonious in its treatment of par coupon rates which in turn are the key driving variable of the prepayment refinancing component.

INTRODUCTION

Mortgage backed securities are assets whose cash flows are derived from the borrower payments of a large number of mortgage loans. A servicing agent collects the payments and after receiving a small fee for the service and after paying any fees due to any agent which has guaranteed principal payments [1], then disburses the remaining cash flows to the investment entity that purchased the loans. The cash flows may be passed through directly as happens for a pool of loans or a pass through security, or they may be cut up in different ways for the convenience of investors. For example, the interest payment may go to one investor (the Interest Only or IO tranche) while the principal payments another (the Principal Only or PO Tranche). For Collateralized Mortgage Obligations or CMOs, the cash flows may be tranched in even more complex ways to satisfy investor needs.

The problem of mortgage backed security valuation goes back many years and creating good MBS valuation models presents many challenges even today. We offer a new approach to pricing mortgage backed securities which not only improves the current state

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1. For example, FNMA and FHLMC both charge a *g-fee* in return for their principal payment guarantee in the event of borrower default.

of the art models, but also offers resolutions to some of the long standing challenges facing MBS valuation models.

For the bulk of derivative products it is standard practice to build valuation and risk measurement models following the replication or risk neutral approach. In fact the study of the actual distribution of possible outcomes of risk variables, whether they are future interest rates, stock prices, FX and so on, does not typically enter the valuation process directly at all. Market participants are perfectly comfortable in the risk neutral world where the risk premium are replaced by higher probabilities of bad outcomes. Just because the yield curve is upward sloping does not mean observed rates are necessarily expected to actually rise. Of course, from a valuation perspective, it is entirely equivalent whether one uses actual probabilities with explicit risk premium or the alternative risk neutral probabilities with no risk premium[2]. However, knowledge of the actual outcomes can nevertheless play an important role in influencing investment choices and other decisions such as hedging strategies.

Yet in the mortgage sector this is not the case. While it is common practice to model the term structure of interest rates following standard no arbitrage risk neutral methods, prepayments have traditionally been treated differently. The study of prepayments is primarily directed at the likely actual outcomes. In part because of their enormous complexity, substantial effort is placed on the econometric study of borrower behavior, which indeed carries its own set of challenges. Most such modeling efforts yield static models in which the projected prepayments are deterministic function of interest rates and other variables such as home prices. Econometric forecasts tend to be more credible in the short run, but the distribution of forecast errors over five or ten years can be substantial. There is, therefore, substantial risk that prepayments will deviate significantly from the model projections and this risk is not being explicitly priced with such an approach. Street practice is to adjust the discounting by an additional spread, the so-called option adjusted spread or OAS in order to correct the pricing. However, since MBS securities are options with payoffs that are non-linear functions of prepayments, ascribing a meaningful interpretation of the OAS spread is not straightforward, especially if one yields to the temptation of viewing it as a metric of excess return. We therefore enhance the standard approach by admitting a distribution of prepayments and in addition by adopting a risk neutral framework [3].

By having a model in which the risks are explicitly leads to a simpler interpretation to market pricing discrepancies. While these errors are reduced, there remains, as always, some residual risks. For example, there can be broadly differing views on prepayments in the market and liquidity issues play a very significant role as well. As for any model, there are inadequacies of the model itself. While the residual spread will not be precisely zero, it is substantially reduced from that found in typical OAS models.

Econometric forecasting models offer enormous value and in fact we use such a model to derive the mean under the physical measure in our stochastic approach. Armed with this mean, we then make appropriate adjustments to the risk neutral measure which

2. See the seminal paper of (**harrison-kreps-79**)

3. See the references at the end of this paper for approaches others have taken

we calibrate to the market prices of liquid assets. In our model we posit that the bulk of the unexplained risk, which presents itself in the OAS of more standard models, is in fact explained by the volatility of prepayments.

While there has been some work done on stochastic prepayments in the literature, (**Kau-04**; **Kolbe-08**) for example, there remain challenges in MBS valuation. One of the flaws of the OAS approach stems from the observation that whenever cash flows are tranched, the sum of the tranche cash flows is identical to the cash flows of the collateral. Therefore, the value of the tranches, ignoring for the time being differential liquidity demand, must add up to the value of the collateral. Furthermore any sensitivity of the value, in other words all greeks, must likewise add up. While the values correctly sum in OAS models by construction, because each tranche has its spread adjusted to match market prices, the sensitivities do not, unless by rare happenstance all the OAS spreads are the same. This flaw is largely resolved in a risk neutral stochastic approach where all the spreads are close to zero.

There is, however, a further difficulty that needs addressing. Prepayment refinancing incentive is driven in part by how much lower available prevailing mortgage rates are than the borrower's current rate. Prevailing par rates, in other words loans that price to par, are not in general known at all times for all states of the world. There are various approaches to modeling the prevailing par coupon rates, both within the dynamics of the valuation model as well as for the projection of the initial par coupon for after shocking rates. While a fully endogenous par mortgage rate would be most satisfying, alternative approaches can be viable as well. In addition to the evolution of mortgage rates within the dynamics, there is the the question of how it is treated under exogenous shocks to the initial parameters. For example, to compute the interest rate DV01, rates are shocked, there is some assumption about the new initial par coupon after the shock. A parsimonious model would price whatever par coupon is assumed under that shock at par. Perhaps the most dramatic issue is how the model reprices change to the initial par coupon rate, sometimes called the mortgage DV01. A reasonable model will at a minimum, under scenario shocks to interest rates or par coupon rates, price the new par coupon asset at par. Typical OAS models fail all these consistency checks.

While there have been some attempts to achieve this consistency, our model offers a parsimonious solution to this problem. Earlier attempts to avoid this problem have varied from ad-hoc adjustments to the price of the OAS to adopting the approach of making the par coupon a complicated but deterministic function of interest rates. In our model we ensure that the initial assumed par coupon indeed prices to par even when the spot interest rates or mortgage rates are shocked. A basis scenario where mortgage rates are shocked while keeping interest rates unchanged is accommodated in our model by an adjustment of the initial value of our stochastic mortgage basis risk variable.

MBS securities are collateralized by large numbers of similar individual mortgage loans and derivative cash flows depend upon the collective prepayments and defaults. Each individual has some probability at each point in time of prepaying or defaulting. A hazard approach is therefore a natural one to model the distributions of cash flows that may ensue. In the event that there are two or more collections which are sufficiently different from one another we can choose different, but correlated, hazard variables for each pool of loans. One can also imagine choosing a small number of independent hazard rate factors with each pool being driven by a combination of these variables. In this paper we consider a single dominant prepayment hazard factor.

Any loan may prepay or default, we therefore can have a pair of variables, one for prepayments and one for defaults. Since a borrower who is likely to default is very unlikely to prepay, the variables are negatively correlated. The credit quality of mortgage pools varies from high, such as those found in prime or Agency pools, to near prime or Alt-A, down to subordinates which have a significant probability of default. The focus of the present work are the prime loans, where defaults are low probability events and can thus be largely ignored. For agency MBS, cash flows are guaranteed and thus a default will be recovered and the balance of the loan will be paid to the investor. The recovery rate is thus very high, representing the high perceived credit of those Government Agencies. These type of defaults are often not distinguished from prepayments in practice or in the econometric analysis of Agency prepayments. Since our focus here will be prime prepayments we will assume the default process to be small enough and, more importantly, has low enough volatility to allow it to be a deterministic adjustment to the mean prepayment, as is common practice.

Even if borrower delinquencies and eventual defaults are low, they can impact the loan guarantor. There is, therefore, additional risk that the guarantor may be impaired in its obligation to cover it's obligations to the investors. The market will price this risk along with other sources of uncertainty such as changes in regulation. Regulatory, political and business changes may impact the agencies in various ways along with additional as yet unknown risks could impact mortgage valuation. For example, it is conceivable that the present-day market could change substantially were agencies to split into a number of smaller companies. All this uncertainty is priced in the market place and in order to accommodate this risk in our valuation model, we should include some way of accounting for. As we will discuss in detail, we do this by following the Occam's Razor principle and introducing a single random variable to serve as a macro proxy for the variety of credit-like risks that the marketplace incorporates in mortgage pricing. For brevity we will refer to this simply as the mortgage risk basis variable.

We describe in section 1 how we formulate the model of stochastic prepayments and defaults along with a mortgage basis variable, keeping our focus on the prepayment risk as discussed above. We will show how we incorporate and extend existing econometric based prepayment models to have stochastic prepayments. We will discuss how we change to the risk neutral measure and calibrate to market prices. We introduce the mortgage risk basis variable in section 2. We put it all together in section 3 where we discuss how to use the model to value mortgage backed securities. We show some results for Agency MBS securities in section 5. In appendix A we show how to calculate the mean and variance of an Ornstein-Uhlenbeck processes which we use in our hazard model.

1. **Incorporating Stochastic Prepayments**

Our goal is to develop a stochastic hazard rate prepayment model for MBS under the risk-neutral measure. Several authors have used this approach including Yevgeny Goncharov (**goncharov-02**) and Stanley Pliska (**pliska-05**) where additional references can be found. We approach the problem in two stages. First we construct the model under the physical measure, which allows us to take full advantage of existing econometric models which will form the core of the drift in our model. For this purpose, rather than appealing to historical data for information about prepayment volatility, we will assume for the moment that it is given. In fact we will use market pricing under the risk neutral measure to calibrate the volatility. Since we are interested in pricing and hedging, it is very important that we derive the volatility from traded prices rather than attempting to extract it from history. Ex post, it may be interesting to compare the market based volatility with historical values.

The econometric approach is deterministic in that given an interest rate path and other economic variables, it projects a unique outcome for prepayments. We choose to make our model consistent with it by interpreting it to be the prepayment under the actual measure. By making the hazard rate process stochastic, we can simulate random shocks that affect the probability of prepayments and defaults. This is a stepping stone to the risk-neutral measure which can then be used for pricing and hedging (**Kau-04**; **Kolbe-08**).

1.1. *Prepayment Hazard rate process*

The cumulative prepayment over one month, also known as the single monthly mortality (SMM), can be expressed as:

$$\text{SMM}\left(t, t+\Delta t, \vec{x}\right) = 1 - e^{-\Lambda(t,t+\Delta t,\vec{x})} \tag{1.1}$$

$$\Lambda\left(t, t+\Delta t, \vec{x}\right) = \int_t^{t+\Delta t} \lambda\left(s, \vec{x}\right) ds \tag{1.2}$$

where Λ is the hazard function, λ is the intensity function , $\Delta t = 1$ month and $\vec{x}$ is a vector of covariates.

Let us define a stochastic intensity function whose dynamics under the physical P-measure follows a mean reverting log normal process.

$$\lambda_t^P\left(\vec{x}\right) \equiv S_t(\vec{x}) h_t^P \tag{1.3}$$

$$dh_t^P = \kappa\left(\theta - h_t^P\right) dt + \sigma\, h_t^P\, dW_t^P \tag{1.4}$$

where h_t is the base line hazard rate function and $S_t(\vec{x})$ is the function associated with explanatory covariates $\vec{x}$. In eq. (1.4) dW_t^P is a Brownian motion under the physical measure P. We note that if the parameters are chosen to satisfy the Feller condition[4] then the hazard rate has bounded variation.

4. $\frac{\sigma^2}{2\kappa} < 1$

The base line function h_t represents a systematic factor that affects all mortgage pools equally.

The function $S_t(\vec{x})$ defines how the covariates $\vec{x}$ affect the prepayment events for each pool. The covariate vector is chosen to be that set of the most relevant parameters suggested by econometric research.

1.2. *Calibration of Baseline Covariate Function*

In our model the instantaneous hazard rate $\lambda_t(\vec{x})$ is not directly observable. What is observed are the cumulative monthly prepayments. We choose our covariate function to be consistent with a given prepayment model. This is done by assuming that the P-measure expected value under of stochastic SMM (1.1) is proportional to the econometric model.

$$E[\text{SMM}_t(\vec{x}, \Delta t)] \equiv E^P\Big[\text{SMM}(\vec{x}, t, t + \Delta t) \mid \{\vec{x}_\tau\}_{\tau \in [0,t]}\Big] \tag{1.5}$$

$$E[\text{SMM}_t(\vec{x}, \Delta t)] = g(t)\widetilde{\text{SMM}}_t(\vec{x}) \tag{1.6}$$

We have freedom of choice when choosing function $g(t)$ in (1.6). If we assume $g(t) = 1$, it would mean that the econometric model fits perfectly the historical SMM time series. Since this is very improbable, we assume that we can use the P-measure mean value of (1.4) as a correction factor.

$$g(t) \equiv E[h_t] = \theta + (h_0 - \theta)e^{-\kappa t} \tag{1.7}$$

Using (1.1) - (1.4) in (1.6), knowing that the expectation of an Itô integral is zero and ignoring terms $O(\Delta t^2)$, we can calculate the expectations conditioned on the realization of the covariate variables, obtaining a formula for the covariate function $S_t(\vec{x})$.

$$S_t(\vec{x}) = -\frac{1}{A_t} \ln\left[1 - E[h_t]\widetilde{\text{SMM}}_t(\vec{x})\right] \tag{1.8}$$

$$A_t = \theta \Delta t + \frac{h_0 - \theta}{\kappa}\, e^{-\kappa t}\left(1 - e^{-\kappa \Delta t}\right) \tag{1.9}$$

where, as is common practice, we have assumed that $S_t(\vec{x})$ is constant over the small interval Δt.

1.3. *Risk Neutral Measure*

The formulation above was all done under the physical measure P. For valuation purposes, we need to transition to the risk neutral measure Q. There are two parts to this measure change. The first comes from the measure change associated with the hazard diffusion equation (1.4), and the second comes from the timing of the hazard event [5].

5. See for example (**shreve-II-10**) for discussion.

The change to the risk neutral measure can be achieved by:

$$dW_t^P = dW_t^Q - \eta(\vec{x}, t)dt \tag{1.10}$$

$$\lambda_t^Q = \mu \cdot \lambda_t^P \tag{1.11}$$

Putting this together with the physical measure dynamics of equations (1.3) and (1.4) leads to the law of motion for the prepament hazard rate under the risk neutral measure. We thus have:

$$\lambda_t^Q(\vec{x}) = \mu S_t(\vec{x}) h_t^Q \tag{1.12}$$

$$dh_t^Q = \tilde{\kappa}\left(\tilde{\theta} - h_t^Q\right) dt + \sigma\, h_t\, dW_t^Q \tag{1.13}$$

with

$$\begin{aligned} \tilde{\kappa} &= \kappa + \sigma\eta \\ \tilde{\theta} &= \frac{\theta}{1 + \sigma\eta/\kappa} \end{aligned} \tag{1.14}$$

The baseline prepayments are captured in $S_t(\vec{x})$ as discussed in section 1.2 and equation (1.8). The additive drift change captures the prepayment risk premium associated with intensity change driven by systematic factors, while the multiplicative adjustment μ is risk compensation due to the prepayment event, i.e., the point process.

While there are alternative parametrizations to these, we adopted the simplest non-trivial choice for the measure change [See (**Kau-04**) and (**Kolbe-08**) for example].

Jarrow et al in (**jarrow-05**) proved that with sufficient diversification, the multiplicative adjustment μ would approach one [6]. The well-diversified portfolio argument assumes that a portfolio contains a countably infinite number of borrowers with independent hazard rate processes. The expected value of the portfolio is invariant to the prepay intensity under the equivalent change of measure as all idiosyncratic risk is diversified away. In our case, the underlying pool of loans is restricted to a small fraction of borrowers in the economy, therefore risk premiums exist associated with prepay or possibly default events. While we expect μ to be greater than 1, it is possible that the market view of expected prepayments may differ from our own model, which could lead to $\mu < 1$.

2. Incorporating Stochastic Mortgage Specific Risk

A model with only stochastic interest rates and prepayments is not sufficient to value mortgage assets. Consider the scenario where there is no new information about prepayments and the rates markets are quiescent and yet mortgages may be trading down. There can be many reasons driving this, new regulatory pressures or potential political impacts or a concern about changes in GSE credit rating or even regulatory capital changes that may make mortgage assets less desirable. These kind of issues suggest the

6. In appendix B we show why $\mu \mapsto 1$ asymptotically.

need for an additional variable to incorporate such mortgage specific risk. It is a common occurrence for mortgage prices to trade up or down in just this way. Rather than trying to identify and model the myriad of possible variables, we adopt a pragmatic and minimalist approach by including an additional variable in our model to explain this market reality. In fact, if one thinks about the par TBA trading lower in price, it is very difficult if not impossible to explain that change in valuation using prepayments alone, even stochastic prepayments. Notwithstanding the fact that such a valuation change has nothing to do with prepayments.

The job of the mortgage basis risk variable ϕ is to account for all the above risks as well as exposure to agency loan guarantee risk which is derived from changes in borrower credit. The law of motion under the risk adjusted measure, with this new variable ϕ, now becomes:

$$\begin{aligned} dh_t^Q &= \tilde{\kappa}\left(\tilde{\theta} - h_t^Q\right)dt + \sigma\, h_t\, dW_t^Q \\ d\phi(t) &= \kappa_\phi\left(\theta_\phi - \phi(t)\right)dt + \sigma_\phi dW_\phi^Q \\ \left\langle dW_t^Q, dW_\phi^Q\right\rangle &= \rho_{h\phi}\, dt \\ \left\langle dZ_t^Q, dW_\phi^Q\right\rangle &= \rho_{r\phi}\, dt \end{aligned} \tag{2.1}$$

The ϕ process initial value is defined *ad hoc* as $\phi(0) \equiv \phi_0 + \beta_m \Delta c_p$. The parameter Δc_p is the mortgage basis and β_m is calibrated such that, in a mortgage basis scenario, the new par coupon is priced at par.

Initial model calibrations suggest that $\rho_{h\phi} = 0$ is a reasonable choice and unless there is a market force which couples these we prefer to leave these uncorrelated. In the next section 3 we'll show how the new variable goes into mortgage valuation.

3. MBS Valuation

Let us consider a mortgage pool with loans following the same hazard process where loan payments are made continuously. Denote interest and scheduled principal payment by c_t and the outstanding notional balance at t under scheduled amortization by B_t. The differential cash flow dCf_t can be expressed as

$$dCf_t = c_t\, e^{-\Lambda^Q(t)}dt + B_t\left(e^{-\Lambda^Q(t)} - e^{-\Lambda^Q(t+dt)}\right) \tag{3.1}$$

where $\Lambda^Q(t) = \int_0^t \lambda^Q(s)ds$ is the hazard function.

Under the risk neutral measure the price of a security is the expected value of the sum of discounted cash flows. Consider the probability space given by $\{\Omega, \mathcal{F}, Q\}$ together with a filtration $\{\mathcal{G}_t : t \geq 0\}$ of sub-σ-fields of $\mathcal{F}$. Now because we have chosen the interest rate and mortgage basis dynamics independent on the prepayment hazard, there is a proper subspace $\{\Omega^h, \mathcal{F}, Q^h\}$ for the prepayment dynamics. The complementary subspace for the non-prepayment variables is given by $\{\Omega^\dagger, \mathcal{F}^\dagger, Q^\dagger)\}$ together with its filtration $\{\mathcal{G}_t^\dagger : t \geq 0\}$ of σ-fields of $\mathcal{F}^\dagger$. The state space $\Omega^\dagger$ is spanned by non-prepayment state variables which are rates and mortgage basis. We can therefore take advantage of this in considering the valuation of a mortgage pool by separating the conditional expected value as follows:

$$\begin{aligned} V_{mbs}(T) &= E^Q\left[\int_0^T e^{-\int_0^t (r_s+\phi_s)\mathrm{d}s}\, dCf_t \mid \mathcal{G}_0\right] \\ &= E^Q\left[\int_0^T e^{-\int_0^t (r_s+\phi_s)\mathrm{d}s}\, e^{-\Lambda^Q(t)}\left(c_t + B_t\,\lambda_t^Q\right)\mathrm{d}t \mid \mathcal{G}_0\right] \\ &= E^{Q^\dagger}\left[E^{\hat{Q^h}}\left[\int_0^T e^{-\int_0^t \left(r_s+\phi_s+\lambda_s^Q\right)\mathrm{d}s}\left(c_t + B_t\lambda_t^Q\right)\mathrm{d}t \mid \mathcal{G}_0\right] \mid \mathcal{G}_0^\dagger\right] \end{aligned} \tag{3.2}$$

where r_t is the instantaneous interest rate.

As described in section 1.1, the hazard rate λ_t is governed by two parts; the function $S_t(\vec{x})$, calibrated to history, and a multiplicative baseline hazard rate h_t, a stochastic process. If you have a deterministic baseline prepayment rate, a constant OAS model solves for a single, constant spread as part of the discount factor to match market price, while our stochastic model matches market price by adjusting parameters associated with λ and the ϕ processes.

If we consider discrete mortgage payments at times $t_i, i = 1, ..., N, \Delta t = t_{i+1} - t_i$, assuming balance B_t is constant between two payment dates, the value of the mortgage pool is given by

$$V_{mbs}(T) = \sum_{i=1}^{N} E^{Q^\dagger}\left[e^{-\int_0^{i\Delta t}(r_s+\phi_s)\mathrm{d}s} E^{Q_h}\left[e^{-\int_0^{i\Delta t}\lambda_s^Q \mathrm{d}s}\left(c_i\Delta t + B_i\lambda_i\Delta t\right) \mid \mathcal{G}_0\right] \mid \mathcal{G}_0^\dagger\right] \tag{3.3}$$

Discretizing eq. (1.1) we get

$$e^{-\int_{(i-1)\Delta t}^{i\Delta t}} = 1 - \mathrm{SMM}_i(\vec{x}) \tag{3.4}$$

$$Su_i(\vec{x}) \equiv e^{-\int_0^{i\Delta t}\lambda_s(\vec{x})\mathrm{d}s} = \prod_{j=1}^{i}\left(1 - \mathrm{SMM}_j(\vec{x})\right) \tag{3.5}$$

where eq. (3.5) is the probability of no-prepayment up to time $i\Delta t$. Using eqs. (3.4) and (3.5) in (3.3) gives us

$$\begin{aligned} V_{mbs}(T) &= \sum_{i=1}^{N} E^{Q^\dagger}\left[e^{-\int_0^{i\Delta t}(r_s+\phi_s)\mathrm{d}s} E^{Q_h}\left[Su_{i-1}\left[\left(1-\mathrm{SMM}_i\right)c_i\Delta t + \mathrm{SMM}_i B_i\right] \mid \mathcal{G}_0\right] \mid \mathcal{G}_0^\dagger\right] \\ &= \sum_{i=1}^{N} E^{Q^\dagger}\left[e^{-\int_0^{i\Delta t}(r_s+\phi_s)\mathrm{d}s} E^{Q_h}\left[Su_i c_i\Delta t + \left(Su_{i-1} - Su_i\right)B_i \mid \mathcal{G}_0\right] \mid \mathcal{G}_0^\dagger\right] \end{aligned} \tag{3.6}$$

where we have omitted the $\vec{x}$ dependency in Su and SMM to simplify the notation.

For multiple pools with independently distributed hazard rates, cash flows across pools are independent, the value of all pools is simply the sum of the value of each pool. Thus the above formula can be employed to price pass throughs and TBAs.[7]

4. Calibration To Market Prices

The model parameters $\{h_0, \mu, \eta, \kappa, \theta, \sigma, \phi_0, \kappa_\phi, \theta_\phi, \sigma_\phi, \rho_{r\phi}\}$ can be calibrated by minimizing an appropriate objective function of the model and market prices. We have chosen a selection of liquid instruments whose prices are well observed in the market. We use a multi-variable search algorithm to find the best parameter set that minimizes the $L2$ norm of the model pricing error:

$$f_{obj} = \sum_{i=1}^{M} \left(V_i^{mkt} - V_i^{model} \right)^2 \tag{4.1}$$

The prices used for calibration are for the most liquid mortgage products: TBAs and IOs. The risk-neutral calibration of the basic model is not capable of matching all the market prices within bid-ask spread. In order to achieve that we need to modify the parametrization of the model to add more degrees of freedom.

One way to achieve this is change some of the constant parameters to be instead a function of the covariates. It is important that this is done in a parsimonious way. In particular the parameters should take the same value for securities whose underlying collateral has the same covariates, otherwise the model could generate internal arbitrage. In particular TBAs and IOs of identical collateral, even if this is purely hypothetical situation, must have identical model parameters.

It is known that the incentive $s = \text{WAC} - m_t$ (m_t is the current mortgage rate) and WALA are two important covariates affecting prepayments and for this reason we have assumed the following

$$\sigma \equiv \sigma_1(s)\sigma_2(\text{WALA}) \tag{4.2}$$

$$\phi_0 \equiv \phi_0(s) \tag{4.3}$$

The functional form of eqs. (4.2) and (4.3) are assumed piece-wise linear and they are bootstrapped from the market prices.

5. Results for Agency MBS

We have calibrated the model to agency MBS prices of 8 TBAs and 10 IOs from October 11^{th} 2012. The simulation had a total of 2000 paths. Some of the model parameters were fixed to decrease the degrees of freedom: $h_0 = 1, \mu = 1$ and $\theta_\phi = 0$.

7. A small modification is needed to price a TBA as TBAs are futures on pass throughs. $V_0^{TBA} = e^{-\int_0^t r_s ds} V_t^{PT}$.

Table 1. Market and model prices for agencies MBS

Type	Coupon(%)	WAC (%)	WALA	Settle	Price	OAS	sOAS
TBA	2.50	3.100	1	14-Nov-12	102.969	-20	0
TBA	3.00	3.600	1	14-Nov-12	105.250	-26	-1
TBA	3.50	4.000	10	14-Nov-12	106.625	-19	-1
TBA	4.00	4.450	23	14-Nov-12	107.281	-6	0
TBA	4.50	4.950	38	14-Nov-12	108.023	3	-1
TBA	5.00	5.650	55	14-Nov-12	108.969	-16	-4
TBA	5.50	6.113	55	14-Nov-12	109.539	-8	-5
TBA	6.00	6.631	50	14-Nov-12	110.539	13	-3
IOSFN-3510	3.50	4.143	23	16-Oct-12	11.281	457	9
IOSFN-4009	4.00	4.552	40	16-Oct-12	10.938	490	54
IOSFN-4010	4.00	4.499	23	16-Oct-12	12.719	384	-20
IOSFN-4509	4.50	4.927	39	16-Oct-12	12.406	537	7
IOSFN-4510	4.50	4.941	26	16-Oct-12	14.344	488	3
IOSFN-5008	5.00	5.639	53	16-Oct-12	10.906	573	51
IOSFN-5009	5.00	5.412	37	16-Oct-12	16.563	481	-38
IOSFN-5010	5.00	5.360	28	16-Oct-12	18.625	422	-62
IOSFN-5508	5.50	6.011	52	16-Oct-12	11.891	645	-14
IOSFN-6008	6.00	6.516	51	16-Oct-12	14.063	544	-10

Table 1 shows that the model fits well the market values. The value $sOAS$ is the remaining OAS after calibration.

[Include graphs of Duration for TBA and IO]

Appendices

Appendix A. Mean and Variance of Mean Reverting Log Normal Process

For a mean reverting log-normal process of the form

$$dh_t = \kappa\,(\theta - h_t)\,dt + \sigma\,h_t\,dW_t \tag{A.1}$$

with W_t a Brownian motion, the mean and variance are computable in closed form. The mean is given by

$$E[h_t] = \theta + (h_0 - \theta)e^{-\kappa t} \tag{A.2}$$

Consider the formal solution to (1.4),

$$h_t = E[h_t] + e^{-\kappa t}\int_0^t e^{\kappa s}\,\sigma h_s\,\mathrm{d}W_s \tag{A.3}$$

from which we have:

$$Var(h_t) = E[e^{-2\kappa t} \int_0^t e^{2\kappa s} \sigma^2 h_s^2 \, \mathrm{d}s] \tag{A.4}$$

$$= e^{-2\kappa t} \int_0^t e^{2\kappa s} \sigma^2 E[h_s^2] \, \mathrm{d}s \tag{A.5}$$

Now, using Itô's lemma

$$dh_t^2 = 2h_t dh_t + \langle dh_t, dh_t \rangle \tag{A.6}$$

$$= (\sigma^2 - 2\kappa) h_t^2 dt + 2\kappa\theta h_t dt + 2\sigma h_t^2 dW_t \tag{A.7}$$

$$d[e^{(2\kappa-\sigma^2)t} h_t^2] = e^{(2\kappa-\sigma^2)t} \left(\left(2\kappa - \sigma^2\right) h_t^2 dt + d[h_t^2] \right) \tag{A.8}$$

$$= e^{(2\kappa-\sigma^2)t} \left(2\kappa\theta h_t dt + 2\sigma h_t^2 dW_t \right) \tag{A.9}$$

from which we find

$$d[e^{(2\kappa-\sigma^2)t} h_t^2] = e^{(2\kappa-\sigma^2)t} \left(\left(2\kappa - \sigma^2\right) h_t^2 dt + d[h_t^2] \right) \tag{A.10}$$

$$= e^{(2\kappa-\sigma^2)t} \left(2\kappa\theta h_t dt + 2\sigma h_t^2 dW_t \right) \tag{A.11}$$

hence

$$e^{(2\kappa-\sigma^2)t} h_t^2 = h_0^2 + \int_0^t e^{(2\kappa-\sigma^2)s} \, 2\kappa \, \theta \, h_s \mathrm{d}s + \int_0^t e^{(2\kappa-\sigma^2)s} \, 2\sigma h_s^2 \mathrm{d}W_s \tag{A.12}$$

From which we immediately get

$$E[h_t^2] = e^{-(2\kappa-\sigma^2)t} E[h_0^2 + \int_0^t e^{(2\kappa-\sigma^2)s} \, 2\kappa \, \theta \, h_s \mathrm{d}s + \int_0^t e^{(2\kappa-\sigma^2)s} \, 2\sigma h_s^2 \mathrm{d}W_s] \tag{A.13}$$

$$= e^{-(2\kappa-\sigma^2)t} \left(h_0^2 + \int_0^t e^{(2\kappa-\sigma^2)s} \, 2\kappa \, \theta \, E[h_s] \mathrm{d}s \right) \tag{A.14}$$

and combining this with equation (A.1) we get

$$E[h_t^2] = e^{-(2\kappa-\sigma^2)t} \left(h_0^2 + \int_0^t e^{(2\kappa-\sigma^2)s} \, 2\kappa \, \theta \, \left(h_0 e^{-\kappa s} + \theta \left(1 - e^{-\kappa s}\right) \right) \mathrm{d}s \right) \tag{A.15}$$

$$= e^{-(2\kappa-\sigma^2)t} \left(h_0^2 + 2\kappa \, \theta \left[\int_0^t \left((h_0 - \theta) \, e^{(\kappa-\sigma^2)s} + \theta e^{(2\kappa-\sigma^2)s} \right) \mathrm{d}s \right] \right) \tag{A.16}$$

$$= e^{-(2\kappa-\sigma^2)t} \left(h_0^2 + \frac{2\kappa\theta \, (h_0 - \theta)}{\kappa - \sigma^2} \left(e^{(\kappa-\sigma^2)t} - 1 \right) + \frac{2\kappa\theta^2}{2\kappa - \sigma^2} \left(e^{(2\kappa-\sigma^2)t} - 1 \right) \right) \tag{A.17}$$

and together with equation (A.4), we find that the variance is given by:

$$Var(h_t) = e^{-2\kappa t} \int_0^t e^{2\kappa s} \sigma^2 E[h_s^2] \, \mathrm{d}s \tag{A.18}$$

$$= e^{-2\kappa t} \Bigg[h_0^2 \left(e^{\sigma^2 t} - 1 \right) \tag{A.19}$$

$$+ \frac{2\theta \, (h_0 - \theta)}{\kappa - \sigma^2} \left(\sigma^2 \left(e^{\kappa t} - 1 \right) - \kappa \left(e^{\sigma^2 t} - 1 \right) \right) \tag{A.20}$$

$$+ \frac{\theta^2}{2\kappa - \sigma^2} \left(\sigma^2 \left(e^{2\kappa t} - 1 \right) - 2\kappa \left(e^{\sigma^2 t} - 1 \right) \right) \Bigg] \tag{A.21}$$

Appendix B. Measure Change for Cox Process in Asymptotic Limit

We show here that in the limit of an asymptotically large number of borrowers, the change from physical to risk neutral measure associated with the counting process and paramatrized by the parameter μ has the property that $\mu \xrightarrow{N \to \infty} 1$. We illustrate the core of the argument here while leaving the discussion of full details to Jarrow et al in (**jarrow-05**).

Consider a simplified one period economy in which N borrowers have taken loans at time t. At a later time $T > t$ if a borrower defaults then the lender receives nothing, otherwise, all being well, they receive a total payment of C from all borrowers. We now assume the pool of borrowers are i.i.d. and then letting Y denote the total income received from all borrowers at time T, we have

$$Y_T = \sum_{i=1}^{N} c_i \, \mathbb{1}_{[i]} \tag{B.1}$$

$$c_i = \frac{C}{N} \tag{B.2}$$

where $\mathbb{1}_{[i]} = 1$ if the i'th borrower has not prepayed at time T. Then we get

$$\begin{aligned} E[Y_T] &= \sum_{i=1}^{N} c_i \, E[\mathbb{1}_{[i]}] \\ &= \sum_{i=1}^{N} c_i \, e^{-\int_t^T \lambda_i \, \mathrm{d}s} \end{aligned} \tag{B.3}$$

Now the variance is,

$$\begin{aligned} Var[Y_T] &= E\left[(Y_T - E[Y_T])^2 \mid t = T\right] \\ &= E\left[\left(\sum_{i=1}^{N} c_i \left(\mathbb{1}_{[i]} - E[\mathbb{1}_{[i]}]\right)\right)^2\right] \\ &= \sum_{i=1}^{N} c_i^2 E\left[\left(\mathbb{1}_{[i]} - E[\mathbb{1}_{[i]}]\right)^2\right] \end{aligned} \tag{B.4}$$

Where the last line follows from the independence assumption which means that

$$E\left[\left(\mathbb{1}_{[i]} - E[\mathbb{1}_{[i]}]\right)\left(\mathbb{1}_{[j]} - E[\mathbb{1}_{[j]}]\right)\right] = 0, \forall i \neq j \tag{B.5}$$

We therefore have

$$\begin{aligned} Var[Y_T] &= \sum_{i=1}^{N} c_i^2 \left(E\left[\mathbb{1}_{[i]}^2\right] - E\left[\mathbb{1}_{[i]}\right]^2\right) \\ &= \sum_{i=1}^{N} c_i^2 \left(E\left[\mathbb{1}_{[i]}\right] - E\left[\mathbb{1}_{[i]}\right]^2\right) \\ &= \sum_{i=1}^{N} c_i^2 \left(e^{-\int_t^T \lambda_i \,\mathrm{d}s} - e^{-2\int_t^T \lambda_i \,\mathrm{d}s}\right) \\ &= \left(\frac{C}{N}\right)^2 \sum_{i=1}^{N} e^{-\int_t^T \lambda_i \,\mathrm{d}s}\left(1 - e^{-\int_t^T \lambda_i \,\mathrm{d}s}\right) \\ &\leq \frac{C^2}{N} \xrightarrow{N \to \infty} 0 \end{aligned} \tag{B.6}$$

In the lim $N \to \infty$ the variance goes to zero and thus we must have $\mu \xrightarrow{N\to\infty} 1$.