ON THIRRING'S APPROACH TO MACH'S PRINCIPLE: CRITICISMS AND SPECULATIONS ON EXTENSIONS OF HIS ORIGINAL WORK*

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ABSTRACT

In his famous paper [1], Thirring obtained inertial forces in the context of linearized Eintein's Relativity Theory. Bass and Pirani, following a suggestion by Lanczos, improved Thirring's calculation [2]. In this work, we criticise Ref.[2] and make some speculations on the main problems of the linear approach. We also compare this classical point of view with a recent Assis' work on Mach's Principle using Weber's law of force [7].

1. Introduction

In the last century, Ernst Mach proposed that inertial forces were caused by interaction of the body with all the great masses of the Universe, the so-called "fixed stars." Einstein named this idea as Mach's Principle and he had the hope that his General Relativity would give a mathematical formulation for it. Therefore there are discussions about Machian effects in General Relativity until nowadays. In Sections 2 and 3 of this work we present the main approaches to Mach's Principle in linearized Relativity; in Section 4 we make a brief comment on non linear approaches; in Section 5 we make our speculations and comments in the linear context and in Section 6 we make comparisons of these approaches with a recent Assis' work on Mach's Principle [7].

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2. Thirring's Approach

In 1918, Hans Thirring [1], derived the inertial forces (Centrifugal and Coriolis) inner a hollow, spinnig, spherical cavity, using linearized equations. The hypothesis that he utilized are the following:

1. the rotation frequency w is constant.
2. $wa \ll 1$ (a the sphere radius and $c = 1$ light's velocity).
3. $||\vec{u}|| \ll 1$ (being $\vec{u}$ the test particle's velocity).
4. $\rho = \rho_0 \delta(r - a)$ (thin shell hypothesis)
5. $g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu}$ (linearized equations).

The momentum-energy tensor for the hollow, spinning cavity utilized was

$$T^{\mu\nu} = \rho_0 v^\mu v^\nu, \tag{1}$$

v^μ being the 4-velocity of an element of the shell and ρ_0 the constant density along it.

Rejecting terms with order higher than $(wa)^2$, he obtained the following equations of motion for a body near the center of the shell

$$\ddot{x} = -\frac{8GM}{3a} w\dot{y} + \frac{4GM}{15a} w^2 x \tag{2}$$
$$\ddot{y} = \frac{8GM}{3a} w\dot{x} + \frac{4GM}{15a} w^2 y \tag{3}$$
$$\ddot{z} = -\frac{8GM}{15a} w^2 z, \tag{4}$$

where M is the rest mass of the shell. Or vectorially,

$$\ddot{\vec{r}} = -\frac{8GM}{15a} w^2 \vec{r} - \frac{4GM}{5a} \vec{w} \times (\vec{w} \times \vec{r}) - \frac{4GM}{3a} (2\vec{u} \times \vec{w}). \tag{5}$$

One feature of the above equations is a radial term which does not have similar in the Newtonian theory. Thirring had already realized it and had conjectured that it was caused by the assumption that the total mass of the Universe could be treated as placed on a shell with infinitesimal thickness. He has guessed that an appropriate distribution of mass could get rid of this term.

3. Bass and Pirani Approach

In 1955, Bass and Pirani [2], following a suggestion made by Lanczos, noted that the energy-momentum tensor given by Thirring did not satisfy the conservation law

$$T^{\mu\nu}{}_{;\nu} = 0. \tag{6}$$

So they have proposed a new energy-momentum tensor that contains an elastic term

$$T^{\mu\nu} = \rho v^\mu v^\nu + E^{\mu\nu}. \tag{7}$$

The terms of $E^{\mu\nu}$ were calculated by analogy with the classical elasticity theory and they are given by

$$E^{\mu\nu} = -\rho\gamma^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & w^2 & iw^3a^2\ \sin^2\theta \\ 0 & 0 & iw^3a^2\ \sin^2\theta & -w^4a^4\ \sin^4\theta \end{pmatrix}, \tag{8}$$

where

$$\gamma = (1 - w^2a^2\sin^2\theta)^{-1/2}, \tag{9}$$

and ρ is the mass density of the shell.

Their hypothesis were the same of Thirring and their calculations were very similar too, but they have made an additional assumption: that the mass density varies with the lattitude according to the equation

$$\rho = \rho_0(1 + Nw^2a^2\sin^2\theta) \tag{10}$$

where ρ_0 and N are constants.

Eq. 10 can be interpreted as a generalization of the Taylor expansion of Eq. 9 when $wa \ll 1$, in the relativistic formula for density:

$$\rho = \gamma^2\rho_0 \tag{11}$$

$wa \ll 1 \Rightarrow \rho \cong \rho_0(1 + \frac{w^2a^2}{2}\sin^2\theta)$ then $N = 1/2$.

Using the tensor (8) in (7) and rejecting terms of order higher than 2 in w, Bass and Pirani obtained the following energy-momentum tensor in cartesian coordinates

$$T^{\mu\nu} = -\rho\gamma^2 \begin{pmatrix} 0 & 0 & 0 & iwa\sin\theta\cos\varphi \\ 0 & 0 & 0 & -iwa\ \sin\theta\cos\varphi \\ 0 & 0 & 0 & 0 \\ iwa\sin\theta\cos\varphi & -iwa\sin\theta\cos\varphi & 0 & 1 \end{pmatrix}. \tag{12}$$

From this tensor the linearized Einstein's theory gives, for a low velocity test particle near the center of the shell, the equations of motion

$$\ddot{x} = -\frac{8GM}{3a}w\dot{y} + \frac{2}{15}(1+N)\frac{GM}{a}w^2 x \tag{13}$$
$$\ddot{y} = \frac{8GM}{3a}w\dot{x} + \frac{2}{15}(1+N)\frac{GM}{a}w^2 y \tag{14}$$
$$\ddot{z} = -\frac{4}{15}(1+N)\frac{GM}{a}w^2 z, \tag{15}$$

or

$$\ddot{\vec{r}} = -\frac{4}{15}(1+N)\frac{GM}{a}w^2\vec{r} - \frac{2}{5}(1+N)\frac{GM}{a}\vec{w}\times(\vec{w}\times\vec{r}) - \frac{4}{3}\frac{GM}{a}(2\vec{u}\times\vec{w}). \tag{16}$$

We see, by comparing the case $N = 0$ with Thirring's equations, that the introduction of the elastic stress has as consequence the halving of the centrifugal force.

The problem of a radial term persists in their work and they did not propose any solution for it as they explicitly state in their paper [2].

4. The Non-Linear Approach

Latter, in 1966, Brill and Cohen studied the effect of rotating masses on inertial frames [3], [4], but they did not use the linear approximation of Relativity Theory. However, they limited their calculations to first order in w and got the correct Coriolis force for the test particle. Recently, this work was extend to second order in w by Pfister and Brawn [5] and they also obtained the centrifugal force for the test particle.

The non-linear approaches do not have the extra radial term appearing in the linear one, but they make the very slowly rotating bodies approximation ($wa \ll 1$). We are not going to give more details on the non-linear approaches since our comments and speculations are restricted to the linear ones.

5. Criticisms and Speculations

We have seen that all approaches make the assumption $wa \ll 1$. But if we want this model of the Universe, this assumption is no more valid, because its radius (a) is extremly big. If we rotate ourselves with $w = 1$Hz, the Moon would have an apparent velocity of 4.0 10^5km/seg; already higher than c.

For this we have extended the work of Bass and Pirani to the case of quickly rotating bodies, that is $wa \gg 1$, where the mass shell would rotate with a velocity much higher than c. For this we use the γ given in Ref.[6]:

$$\gamma = (w^2a^2\sin^2\theta - 1)^{-1/2} \qquad \text{and} \quad wa \gg 1. \tag{17}$$

We do not see any contradiction between the assumption $wa \gg 1$ and the General Relativity. The superluminal mass shell would be in eternal and constant motion in our model, so it could not transmit information.

A justification for using linearized equations is that, in common life situations, the centrifugal force can be of the order of our weight, and we know that Earth's field is a weak field; as the centrifugal force in these situations.

We observe that Bass and Pirani have used ρ instead of ρ_0 in their $T^{\mu\nu}$ (Eq. 7), what is not correct because the velocity dependence of the density is already implicit in the definition of the energy-momentum tensor (from the 4-velocity). However, this is not a problem in their work, since for small w an appropriate choice of the constant N maskes the mistake. The correct energy-momentum tensor would then be

$$T^{\mu\nu} = \rho_0 v^\mu v^\nu + E^{\mu\nu}. \tag{18}$$

Applying Eq. 8 in Eq. 18 and only keeping terms in $1/wa$ and $(1/wa)^2$ (we have supposed $wa \gg 1$) we get the following superluminal tensor in cartesian coordinates,

$$T^{\mu\nu} = -\rho_0 \begin{pmatrix} 0 & 0 & 0 & iwa\sin\theta\sin\varphi \\ 0 & 0 & 0 & -iwa\sin\theta\cos\varphi \\ 0 & 0 & 0 & 0 \\ iwa\sin\theta\sin\varphi & -iwa\sin\theta\cos\varphi & 0 & w^2a^2\sin^2\theta \end{pmatrix}. \tag{19}$$

If we use Eq. 10 in Eq. 12 and reject terms with order higher than second in w, we get the same tensor as Eq. 19. So it is not surprising when we repeat the calculations and obtain that the test particles's motion equation for $wa \gg 1$ are the same of Bass and Pirani with $N = 0$ ($\rho = \rho_0$),

$$\ddot{\vec{r}} = -\frac{4GM}{15a} w^2 \vec{r} - \frac{2GM}{5a} \vec{w} \times (\vec{w} \times \vec{r}) - \frac{4GM}{3a} (2\vec{u} \times \vec{w}) \tag{20}$$

The radial term also appears in this approach. The fact that in the Newtonian theory the field inside a massive sphere is radial but is zero inside a shell suggests that the thin shell model should be the responsable for the appearence of an extra radial term. It seems that this was Thirring's point of view, since he stated in his paper that the radial term should arise from the mass shell approximation. (See Ref.[1])

Bass and Pirani did not get any solution for the radial term problem, and we believe that it is a consequence of the linear approximation, since it does not appear in the non-linear approach [5].

6. Comparison with Assis' Work

In 1989, Assis proposed an equation for gravitation in analogy with the Weber's force law for Electrodynamics [7]. This had already been proposed by Tisserand [8] but not yet applied in connection with Mach's Principle as Assis did.

The force exerted by the particle j on i is given by

$$\vec{F}_{ji} = -H_g \frac{m_i m_j}{r_{ij}^2} \hat{r}_{ij} \{1 + \frac{\xi}{c^2} (\vec{v}_{ij} \cdot \vec{v}_{ij} - \frac{3}{2} (\hat{r}_{ij} \cdot \vec{v}_{ij})^2 + \vec{r}_{ij} \cdot \vec{a}_{ij})\} \tag{21}$$

where the ij index is for the relative position, velocity and acceleration; H_g and ξ are constants.

In his work, Assis fixes the value $\xi = 6$ to fit the experimental advance of the perhelion of Mercury, but the constant H_g remains undefined.

From Eq. 21, Assis calculated the force that a spinning, massive sphere with isotropical mass distribution exerts in a particle that is inside it (not necessarily near its center), obtaining the equation of motion

$$\ddot{\vec{r}}_p = -\Phi \left[\vec{a}_p + \vec{r}_p \times \frac{d\vec{w}}{dt} + 2\vec{v}_p \times \vec{w} + \vec{w} \times (\vec{w} \times \vec{r}_p) \right], \tag{22}$$

where

$$\Phi = \frac{H_g \xi M}{2ac^2} \tag{23}$$

($\vec{a}_p, \vec{v}_p, \vec{r}_p$ are the acceleration, velocity and position of the particle; $\vec{w}$ the angular velocity of the spinning shell, a is its radius, and M is its mass).

Taking $\vec{w}$ constant, as did Thirring,

$$\ddot{\vec{r}}_p = -\Phi \vec{a}_p - \Phi \vec{w} \times (\vec{w} \times \vec{r}_p) - \Phi (2\vec{v}_p \times \vec{w}) \tag{24}$$

Comparing Eq. 24 with Eq. 20 we observe that

1. In Eq. 24 there is no radial term as in Eq. 20.
2. In Eq. 24 there is an acceleration dependent term. It is not present in Eq. 20, since the inertia is already included in Minkowsky metric and Eq. 20 is a result of a perturbation of it, keeping only perturbation terms.
3. In Eq. 24 the constants multiplying the Coriolis and centrifugal forces are the same while in Eq. 20 we have different constants for this terms. This is probably the influence of the radial term that has spoiled the centrifugal force's constant.

By comparing Eq. 20 and Eq. 24 we intend to relate H_g with the universal gravitation constant G. For this we make Φ equal to the constant appearing in the Coriolis force, because from (iii) this is the correct procedure. We get

$$H_g = \frac{8G}{3\xi} \qquad \text{if} \quad \xi = 6, \quad H_g = \frac{4G}{9} \tag{25}$$

and H_g is of the order of G as should be expected.

7. Conclusion

In this work we have analysed the method of Thirring to obtain the inertial forces inside a hollow, spinning sphere. The appearance of a radial term that does not exist in classical mechanics seems to be a consequence of the linear approximation and not a consequence of the thin shell approximation as Thirring first thought.

We observe that if we consider the spinning sphere as a model of the Universe, the hypothesis $wa \ll 1$ is no more valid. So we have done an extension of Thirring's approach for $wa \gg 1$, obtaining that the equation of motion is the same in both limits as should be expected.

Finally, we have made a comparision of these classical approaches and a recent work of Assis on Mach's Principle using a force equation from Weber's electrodynamics, calculating an unkown constant H_g of his theory from the universal gravitation constant G.

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