Valuation of a Homogeneous Collateralized Debt Obligation

by

Fabio Mibielli Peixoto [1]

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Waterloo, Ontario, Canada, 2004

[1] E-mail: fabiop@aya.yale.edu
[2] Updated in August 2005.

Abstract

Monte Carlo simulation and a semi-analytical method are used to value a basket default swap and an homogeneous Collateralized Debt Obligation (CDO). The semi-analytical technique is based on the one factor copula model proposed by J.P. Laurent and J. Gregory [1]. We study the properties of a CDO with Monte Carlo and compare the spread calculation with the one obtained by the factor model.

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Chapter 1

Introduction

In the last twenty years one of the main innovations in the field of Finance has been securitization. This is the process of pooling together a portfolio and issuing liability and equity notes backed by this pool of assets. It started in the late 1970s with mortgages backed securities and it has expanded to other instruments like credit card debt, student and car loans, junk bonds, etc. Its main purpose for the originator is to transfer some of the risk to the investors and to free up regulatory capital. The main advantage for the investors is diversification: they are able to invest in products that they would not have access otherwise.

In this report we will study one of the main examples of securitization: a Collateralized Debt Obligation (CDO). This is a security that has had a tremendous growth in the last couple of years linked to the surge of the credit derivatives market. Using Monte Carlo simulation and a semi-analytical method we analyze and find the no-arbitrage price of a CDO. We explain both methods in details and compare their results. The semi-analytical method is based on the single factor copula model proposed by J.P. Laurent and J. Gregory [1] and recently used also by J. Hull and A. White [3].

In this introduction we define and classify a CDO explaining how it works. In chapter 2 we make a review of the mathematical tools necessary to study a defaultable portfolio. In chapter 3 we apply Monte Carlo simulation to study a CDO and in chapter 4 we explain and use the single factor copula model to price a CDO. In chapter 5 we present our conclusion.

1.1 Definition and Classification

A CDO is an asset backed security whose collateral consists mainly of a portfolio of defaultable instruments like loans, junk bonds, mortgages, etc. If the portfolio contains only credit default swaps (CDS), it is called a synthetic CDO. This has become one of the most popular CDOs in the last couple of years. The reason is that with a CDS the bank can create a portfolio with exposure to defaultable instruments without the requirement of owning them, facilitating their creation.

The main economic reason for the existence of CDOs is to address market imperfections. It frees regulatory capital for financial institutions and it improves liquidity for some bonds and loans. For investors it creates risk-return profiles hard to achieve in other ways and it gives them opportunity to invest in less liquid assets.

The issuance of CDOs has had an exponential increase for the last 10 years (see Fig. 1.1). This sudden growth is due to its better acceptance by investors, especially hedge funds, and credit risk hedgers [2]. Also the surge of the credit derivatives market in the last couple of years has contributed to the increase in the number of synthetic CDOs.

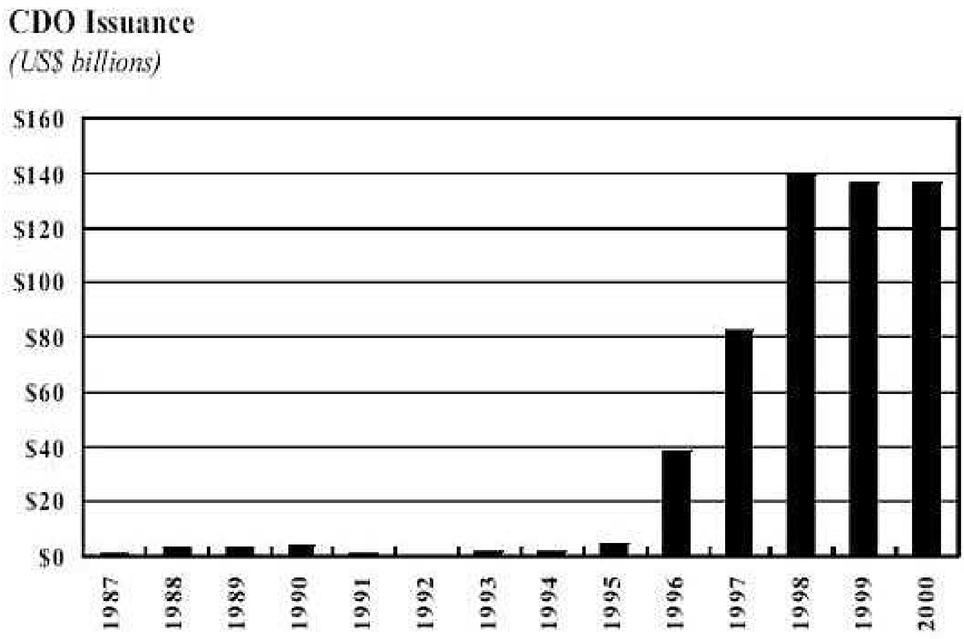

Figure 1.1: Estimated global number of CDOs from 1987 to 2000. Source: JP Morgan [2].

CDOs can be classified in two categories: arbitrage and balance sheet (see Fig. 1.2). An arbitrage CDO is setup to make money from the difference between the cost of acquiring the collateral portfolio and the sale of the notes. A balance sheet CDO has the purpose of freeing up regulatory capital that is tied up to the collateral asset. In this case the portfolio consists mainly of loans [4].

We can further divide CDOs into cash flow and market value types. A cash flow CDO is not subjected to active trading by the manager and its main source of risk is due to credit default. A market value CDO has its collateral portfolio marked-to-market frequently, therefore it has an additional risk linked to the performance of the CDO manager. In this report we will analyze only cash flow CDOs, since nine out of ten CDOs, both by number and by volume, are of this type [2].

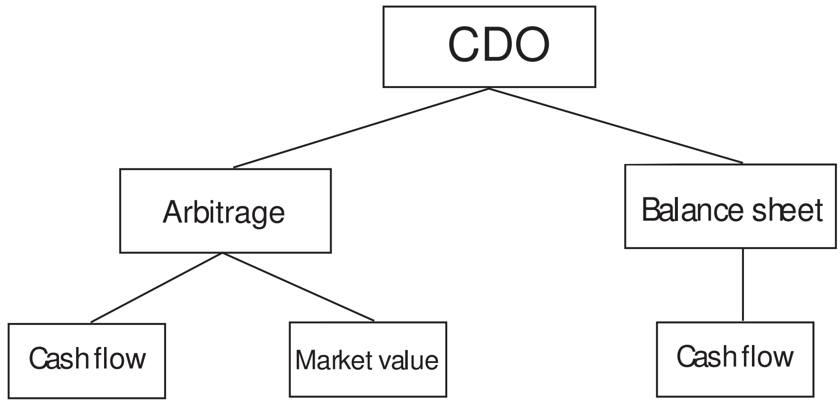

Figure 1.2: General classification of CDOs.

1.2 How a CDO works

A Special Purpose Vehicle (SPV) is setup to hold the portfolio of assets. It issues notes with different subordination, so called tranches, that is sells to investors (see Fig. 1.3). The principal payment and interest income (LIBOR + spread) are allocated to the notes according to the following rule: senior notes are paid before mezzanine and lower rated notes. Any residual cash is paid to the equity note. The equity note is also called the first-loss position because it is the first to be affected by a default in the portfolio, offering a larger spread than the more senior notes. In practice we can say that the CDO investors are selling protection, choosing a tranche according to their risk - return preference.

A consequence of this payment subordination is that a senior note is less risky than the collateral. As we move down the CDO's notes, the risk level increases. Each tranche, except for the equity, is rated by credit rating agencies. The senior notes usually have Aaa/AAA rating. The originating bank has an incentive to keep a significant share of the equity tranche to demonstrate its commitment to the CDO

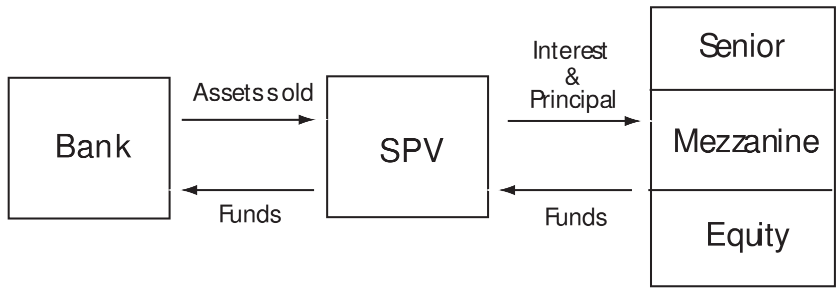

Figure 1.3: Schematics of how a CDO is setup.

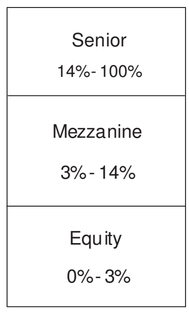

Figure 1.4: Capital structure of a simple CDO.

performance.

As an example consider a CDO with three tranches: equity, mezzanine and senior. Suppose that the total CDO notional is 100 million and the capital structure is the one shown in Fig. 1.4. During the CDO lifetime some instruments in the collateral portfolio might default. If at maturity the total default loss is less than 3 million, only the equity tranche is affected. If it is between 3 and 14 million, the equity tranche does not get the principal back and the mezzanine gets only part of it. If the loss is more than 14 million than the equity and mezzanine do not get anything back and the senior tranche gets whatever is left.

Chapter 2

Mathematical Background

In this chapter we will present the basic mathematical tools to study the default time of a credit portfolio. We assume a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, where Ω is the set of all states, $\mathcal{F}$ is a σ-field representing the collection of all events, $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration representing the increase in information with time and $\mathbb{P}$ is a probability measure. When pricing an instrument we always assume no-arbitrage and in this case $\mathbb{P} = \mathbb{Q}$ is the risk-neutral measure.

2.1 Stopping Time

When studying a credit portfolio, the first thing we have to model is the time of default τ for each instrument. It should be a random variable taking values in $\mathbb{R}^+ \cup \{\infty\}$, where we include $\{\infty\}$ to take into account a no-default event. Also, if we assume a market with free flow of information, the event of default should be known to everyone when it occurs. Therefore τ is not only a random time but also a stopping time with respect to $\{\mathcal{F}_t\}_{t\geq 0}$:

$$\{\tau < t\} \subset \mathcal{F}_t, \ \forall t \geq 0, \tag{2.1}$$

where $\{\tau < t\}$ is the inverse image of $\tau(\omega) < t$. Notice that the requirement that τ is a stopping time does not mean that the default event must come as a surprise. The concept of a predictable stopping time can be used for predictable defaults [5].

For modelling purposes we can represent the default event at time t as:

$$N_\tau = \mathbb{I}_{\{\tau \leq t\}}, \tag{2.2}$$

where $\mathbb{I}$ is the indicator function ($\mathbb{I}_{\{\tau\leq t\}} = 1$ if $\tau \leq t$, $\mathbb{I}_{\{\tau\leq t\}} = 0$ if $\tau > t$).

2.2 Hazard Rate

The hazard rate function h plays an important role in intensity models for credit risk. Intuitively a large value of h means a higher probability of default. Its formal definition is the following [5].

Definition 2.2.1 *Let τ be a stopping time and $F(t) = \mathbb{P}(\tau \leq t)$ its distribution function. Assuming that $F(t) < 1$ for all t and that $f(t)$ is the density function, then the hazard rate h of τ is given by:*

$$h(t) = \frac{f(t)}{1 - F(t)} = \frac{1}{1 - F(t)}\frac{dF(t)}{dt}. \tag{2.3}$$

Using $\Delta F = P(t < \tau \leq t + \delta t, \tau > t)$ and $1 - F(t) = P(\tau > t)$ in (2.3) we can write:

$$h(t) = \lim_{\delta t \to 0} \frac{P(t < \tau \leq t + \delta t | \tau > t)}{\delta t}. \tag{2.4}$$

Equation (2.4) gives us an interpretation of $h(t)$. It is the probability of default, per unit of time, in $(t,\ t+\delta t)$, given that no default happened before t. From Eq.(2.3) we can re-write $F(t)$ as:

$$F(t) = 1 - e^{-\int_0^t h(s)ds}\ , \tag{2.5}$$

$$S(t) = 1 - F(t) = e^{-\int_0^t h(s)ds}\ , \tag{2.6}$$

where $S(t) = P(\tau > t)$ is called the survival function.

2.3 Point Processes

A single default time is represented by a stopping time as seen in section 2.1. The generalization for multiple default events constitutes a point process.

Definition 2.3.1 *A point process is an increasing sequence of stopping times* $\{\tau_1,\ \tau_2,\ \ldots\}$ *where* $0 < \tau_i < \tau_{i+1}\ \forall\ i \in \mathbb{N}$.

A point process is a good way to represent a sequence of defaults in a credit portfolio or multiple defaults of a single obligor. The total number of defaults is represented by a counting process:

$$N(t) = \sum_{i=1}^{n} \mathbb{I}_{\{\tau_i \leq t\}}, \tag{2.7}$$

$$N(0) = 0. \tag{2.8}$$

This is a stochastic process with natural filtration $\mathcal{G}_t = \sigma\{N(s),\ 0 \leq s \leq t\}$. More specifically $N(t)$ is a submartingale as shown below:

$$E[N(s)|\mathcal{G}_t] = \sum_{i=1}^{n} E[\mathbb{I}_{\{\tau_i \leq s\}}|\mathcal{G}_t],\ s > t > 0, \tag{2.9}$$

$$\mathbb{I}_{\{\tau_i \leq s\}} = \mathbb{I}_{\{\tau_i \leq t\}} + \mathbb{I}_{\{t \leq \tau_i \leq s\}}, \tag{2.10}$$

$$E[N(s)|\mathcal{G}_t] = N(t) + \sum_{i=1}^{n} P(t \leq \tau_i \leq s|\mathcal{G}_t), \tag{2.11}$$

$$E[N(s)|\mathcal{G}_t] \geq N(t). \tag{2.12}$$

We can then apply the Doob-Meyer decomposition theorem to the counting process $N(t)$ [6].

Theorem 2.3.1 (Doob-Meyer decomposition) *Let $X(t)$ be a right-continuous, non-negative submartingale with respect to $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Then there exists a right-continuous martingale $M(t)$ and an increasing, right-continuous process $A(t)$ such that $E[A(t)] < \infty$ and $X(t) = M(t) + A(t)$ a.s. for any $t \geq 0$. If we assume that $A(t)$ is predictable, this decomposition is unique.*

The theorem above implies that there exists a unique, predictable process $A(t)$ such that $\{N(t) - A(t)\}_{t\geq 0}$ is a right-continuous $\mathcal{F}_t$-martingale. The process $A(t)$ is called the compensator of $(N(t),\ \mathcal{F}_t)$. Its importance is that it gives some information about the probability of defaults over the next period of time:

$$E[N(t+\delta t) - N(t)|\mathcal{F}_t] = E[A(t+\delta t) - A(t)|\mathcal{F}_t], \tag{2.13}$$

since $\{N(t) - A(t)\}_{t\geq 0}$ is an $\mathcal{F}_t$-martingale.

Definition 2.3.2 *If $A(t)$ is differentiable, we can write $A(t) = \int_0^t \lambda(s)ds$, where $\lambda(t)$ is a positive, predictable process called the stochastic intensity of $N(t)$.*

Credit models where $A(t)$ is differentiable are called intensity models. We can relate $\lambda(t)$ to the probability of one default at time t, given the information at time t^{-}:

$$E[dN(t)|\mathcal{F}_{t^-}] = dA(t), \tag{2.14}$$

where we have used (2.13) and the fact that $A(t)$ is predictable. Assuming that $dN(t) = \{0,\ 1\}$ in a short period of time we have:

$$P[dN(t) = 1|\mathcal{F}_{t^-}] = \lambda(t)dt. \tag{2.15}$$

If we use the natural filtration $\mathcal{G}_t = \sigma\{N(s),\ 0 \leq s \leq t\}$ and some regularity conditions, we can show that the intensity process is equal to the hazard rate [5]:

$$\lambda(t) = h(t). \tag{2.16}$$

2.4 Distribution of Default Times

In this report we will consider only the simple case of a piece-wise constant hazard rate. This is a simplifying approximation that is actually used quite often in the financial industry. We will also assume that N(t) follows a homogeneous Poisson process with intensity $\lambda = h$.

Definition 2.4.1 *The counting process $\{N(t)\}_{t\geq 0}$ follows a homogeneous Poisson process with intensity λ if and only if:*

- $\{N(T) - N(t)\}_{T>t\geq 0}$ *are stationary and independent increments.*
- $N(t)$ *has a Poisson distribution with parameter* λ*.*

$$P[N_T - N_t = k] = \frac{1}{k!}[\lambda(T-t)]^k e^{-\lambda(T-t)}\ . \tag{2.17}$$

Following Lando's paper [7] we can simulate the first time of default τ. Assuming $N_0 = 0$, we have from (2.17) the probability of no-default:

$$P(N_t = 0) = e^{-\lambda t}. \tag{2.18}$$

The survival function $S_\tau(t) = P(\tau > t)$ and the distribution function $F_\tau(t) = P(\tau \leq t)$ of the first time of default are:

$$S_\tau(t) = e^{-\lambda t}, \tag{2.19}$$

$$F_\tau(t) = 1 - e^{-\lambda t}. \tag{2.20}$$

Using the inverse transform method on (2.20) we can simulate τ:

$$\tau = \frac{-\ln U}{\lambda}, \tag{2.21}$$

where U is a uniform random variable in $[0, 1]$. We can apply Eq.(2.21) to simulate the time of default for each instrument in a credit portfolio. This will be useful in chapter 3.

2.5 Copula Functions

When analyzing a portfolio with n defaultable instruments, one of the most important parameters to know is the joint default distribution:

$$F(t_1, \ldots, t_n) = P(\tau_1 \leq t_1, \ldots, \tau_n \leq t_n). \tag{2.22}$$

Finding out the format of F is not an easy task due to the fact that different joint probability distributions can have the same marginal functions. The connection between the two is given by the copula function. It separates the structural dependence of the random variables from their marginal behavior [8], [9].

Definition 2.5.1 *An n-dimensional copula function* $C: [0,1]^n \to [0,1]$, $n \in \mathbb{N}$ *is a joint distribution function with uniformly distributed marginal on* $[0,1]$.

$$C(u_1, \ldots, u_n) = P(U_1 \leq u_1, \ldots, U_n \leq u_n), \tag{2.23}$$

$$U_i \sim U[0,1],\ i = \{1, \ldots, n\}. \tag{2.24}$$

The most important theorem in the theory of copulas is Sklar's theorem [10], [11]. It proves that copulas are the link between the marginal functions and the joint distribution function.

Theorem 2.5.1 (Sklar) *Let* F *be a joint distribution function on* $\mathbb{R}^n$ *with marginal distribution* F_i, $i = \{1, \ldots, n\}$. *Then there exists a copula function* C *such that* $\forall (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)). \tag{2.25}$$

If the F_i*s are continuous, the copula is unique.*

Corollary 2.5.1 *Let* F *be an n-dimensional joint distribution function with marginal distribution* F_i, $i = \{1, \ldots, n\}$. *An n-dimensional copula* C *can be built as:*

$$C(u_1, \ldots, u_n) = F[F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)], \tag{2.26}$$

where F_i^{-1} *is the quasi-inverse function defined by* $F_i^{-1}(u_i) = inf\{x | F_i(x) \geq u_i\}$.

The corollary above is the key to generate many different copulas. One of the first to be used in credit risk was the Gaussian copula [12]. It is obtained by using the multivariate normal distribution Φ_Σ (Σ is the correlation matrix) and the normal marginal distribution Φ in (2.26):

$$C_{Gauss}(u_1, \ldots, u_n) = \Phi_\Sigma[\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)]. \tag{2.27}$$

If we assume as in [12] that the dependence structure in a credit portfolio is represented by a Gaussian copula, the joint probability distribution of default is obtained by using (2.27) in (2.25):

$$P(\tau_1 \le t_1, \ldots, \tau_n \le t_n) = \Phi_\Sigma[\Phi^{-1}(F_1(t_1)), \ldots, \Phi^{-1}(F_n(t_n))], \tag{2.28}$$

where $F_i(t_i)$ is given by (2.5). Notice that each default time τ_i can have a different marginal distribution.

There are many other copulas that can be used in credit risk besides the Gaussian, including the t-copula, the Clayton copula, etc [8], [9]. However, because we will not study the CDO price sensitivity to the copula choice, we will always assume a Gaussian copula in this essay.

Chapter 3

Monte Carlo simulation of a CDO

Consider a portfolio with n instruments, each one with notional N_i, $i = \{1, \ldots, n\}$. We associate a random variable τ_i to each one of them representing their default time with marginal default distribution:

$$F_i(t) = P(\tau_i \leq t),\ t \geq 0. \tag{3.1}$$

The default event correlation between instruments i and j is given by:

$$\rho_{ij}(t) = corr\left(\mathbb{I}_{\{\tau_i<t\}}, \mathbb{I}_{\{\tau_j<t\}}\right), \tag{3.2}$$

where $\mathbb{I}$ is the indicator function.

The default loss for instrument i is a random variable represented by:

$$L_i(t) = (1 - R_i)N_i\mathbb{I}_{\{\tau_i<t\}}, \tag{3.3}$$

where R_i is the recovery rate, assumed deterministic in our case. Therefore the total portfolio loss is given by:

$$L(t) = \sum_{i=1}^{n} L_i(t). \tag{3.4}$$

In the case of a CDO we are interested in knowing how the total loss (3.4) affect a tranche with a lower bound $a\%$ and an upper bound $b\%$. From the payment subordination we know that the tranche [a, b] suffers a loss at time t if and only if:

$$a\%N_T < L(t) \leq b\%N_T, \tag{3.5}$$

where $N_T = \sum_{i=1}^{n} N_i$ is the total portfolio value. Due to condition (3.5) we can represent the tranche loss $L_{A,B}(t)$ as:

$$L_{A,B}(t) = [L(t) - A]\,\mathbb{I}_{\{L(t)\in[A,B]\}} + (B - A)\mathbb{I}_{\{L(t)\in[B,N_T]\}}, \tag{3.6}$$

where $A = a\%N_T$ and $B = b\%N_T$.

3.1 Expected Loss

The first property of the CDO that we will analyze is the expected loss for each tranche at maturity. Using Monte Carlo simulation and Eq.(3.6) this becomes a straightforward task. For this purpose we are going to assume a simple structure as the one depicted in Fig. 1.4. Moreover we want to focus on the dependence of the expected loss on the correlation and recovery rate, hence we will simplify our model assuming a homogeneous CDO: same notional N, recovery rate R and hazard rate h for all obligors. We also use a constant correlation matrix with only one parameter (a flat correlation):

$$\rho_{ij} = \rho,\ i \neq j, \tag{3.7}$$

$$\rho_{ij} = 1,\ i = j, \tag{3.8}$$

$$0 < \rho < 1. \tag{3.9}$$

This is a major assumption, but it is a very common one in the industry. A one dimensional correlation matrix can be treated in a similar way as implied volatility in the Black-Scholes formalism.

We will study a portfolio with $n = 100$ instruments, maturity T = 5 years and we assume that the risk-free interest rate r is constant at 5% p.a. Using these assumptions we can simplify (3.5) and (3.6):

$$a < \mathcal{L}(t) \leq b, \tag{3.10}$$

$$\mathcal{L}(t) = L(t)/N. \tag{3.11}$$

The tranche loss is given by:

$$\frac{L_{a,b}(t)}{N} = [\mathcal{L}(t) - a]\, \mathbb{I}_{\{\mathcal{L}(t)\in[a,b]\}} + (b-a)\mathbb{I}_{\{\mathcal{L}\in[b,100]\}}. \tag{3.12}$$

We use a Gaussian copula to model the dependence structure in the portfolio. The procedure to estimate the expected loss at maturity with Monte Carlo simulation is the following [7], [13]:

1. Generate n multinormal random variates v_i with mean zero and correlation matrix given by (3.7) and (3.8).

2. Assuming a constant hazard rate h for n obligors, the default time is given by (2.21):

   $$\tau_i = \frac{-\ln u_i}{h},\ i = 1, \ldots, n\ , \tag{3.13}$$

   where $u_i = \Phi(v_i)$ and Φ is the standard normal distribution.

3. Using (3.12) and (3.13) we calculate the loss at maturity $L^k_{a,b}(T)$.

4. Repeat procedure 1 to 3 m times. The estimator for the expected loss is:

   $$E[L_{a,b}(T)] = \frac{1}{m} \sum_{k=1}^{m} L^k_{a,b}(T). \tag{3.14}$$

Table 3.1 shows the value and the standard error for the expected loss calculated with 50,000 iterations, $h = 0.03$, $\rho = 0.3$ and R = 0.4. Notice that crude Monte Carlo is already good enough to give us a relative error less than 2%.

Tranche	Exp. Loss (%)	Std. error (%)
14% - 100%	1.83	0.02
3% - 14%	39.23	0.18
0% - 3%	82.59	0.14

Table 3.1: Expected loss for each CDO tranche calculated with 50,000 iterations, h = 0.03, $\rho = 0.3$ and R = 0.4.

Figure 3.1 shows the dependence of the expected loss at maturity on correlation with fixed recovery rate. The position of an investor that buys an equity or mezzanine tranche is “long” correlation: larger the correlation in the future, smaller will be his expected loss. An investor who buys a senior tranche is “short” correlation: he is betting that correlation will remain small or go down in the future. This terminology is commonly used in the industry [14].

Figure 3.2 shows the dependence of the expected loss with recovery rate for a fixed correlation ($\rho = 0.3$). All tranche losses decrease with higher recovery rate as expected.

Figures 3.3 and 3.4 show the loss distribution of a portfolio with different recovery rates and fixed correlation. Notice that the one with larger recovery rate (Fig. 3.4) has a smaller right tail, meaning that the senior tranche gets less affected by the losses. This can also be observed in Fig. 3.2 where the senior expected loss gets close to zero at high recovery rates.

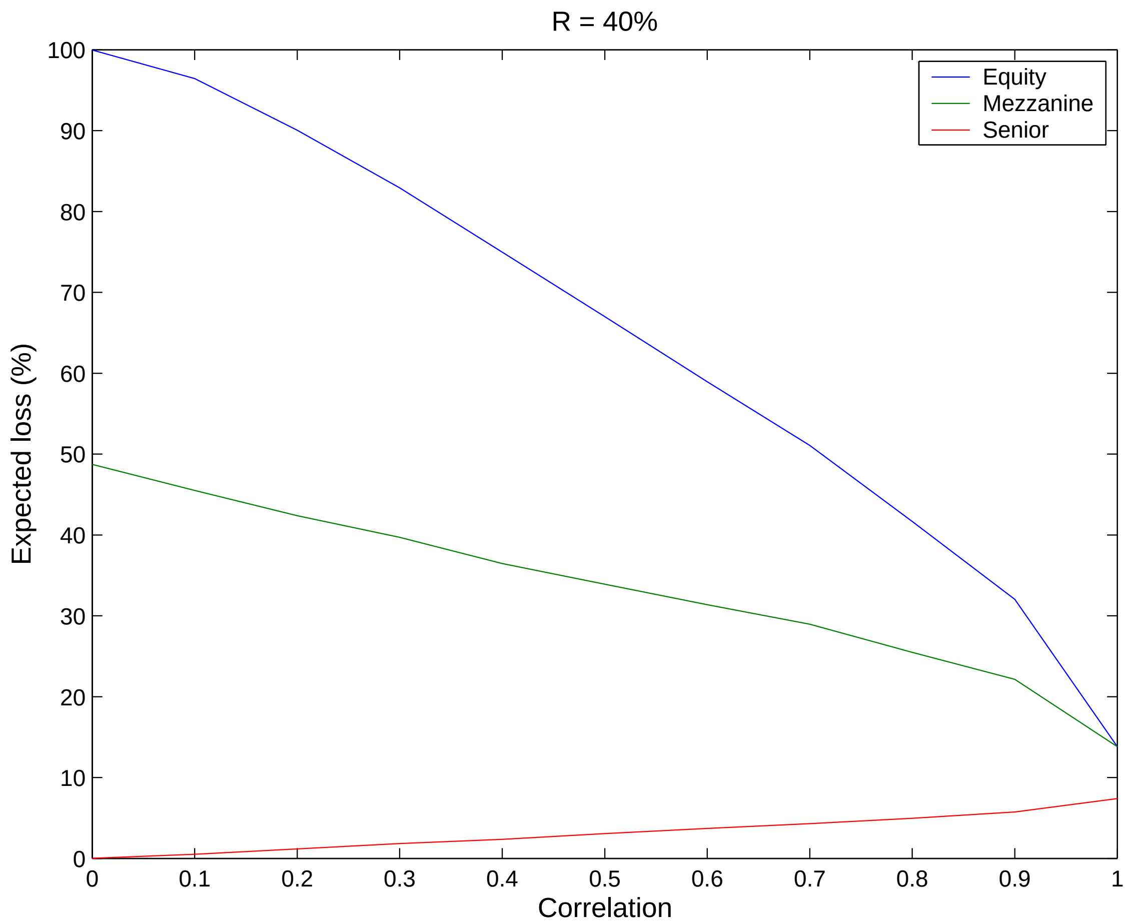

Figure 3.1: Dependence of expected loss with correlation, for 50,000 iterations, h = 0.03 and R = 0.4.

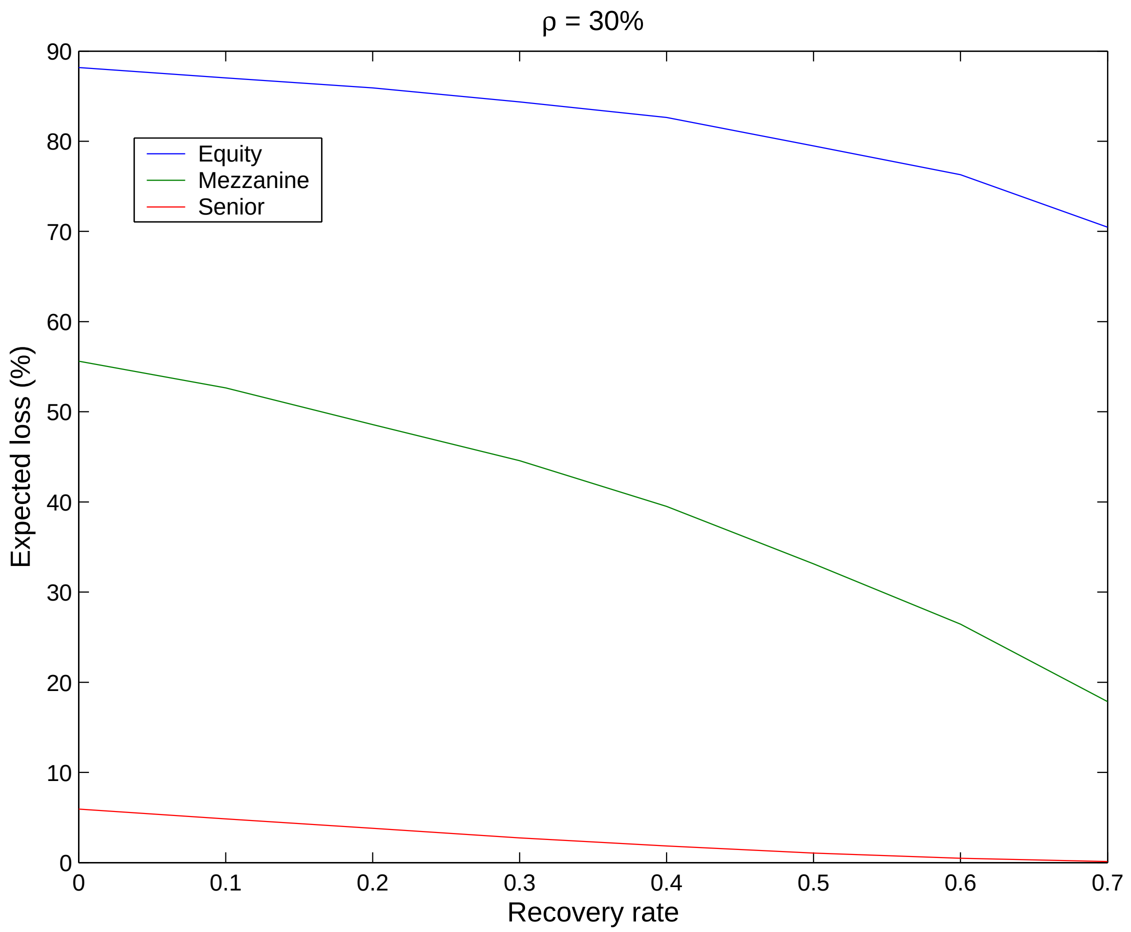

Figure 3.2: Dependence of expected loss with recovery rate for 50,000 iterations, h = 0.03 and $\rho = 0.3$.

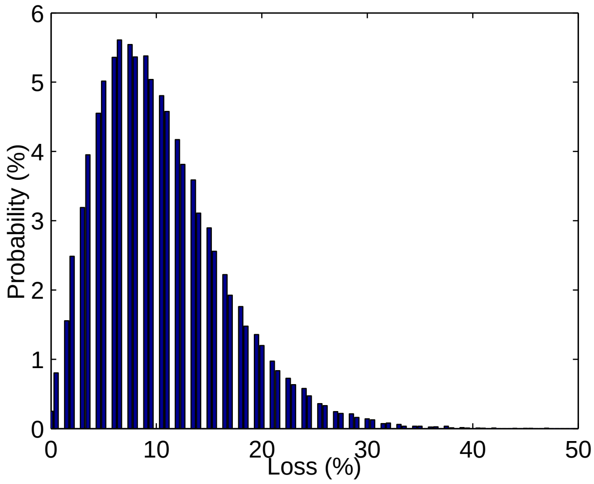

Figure 3.3: Portfolio loss distribution for correlation at 10%, recovery rate 25%, hazard rate 0.03 and 50,000 simulations.

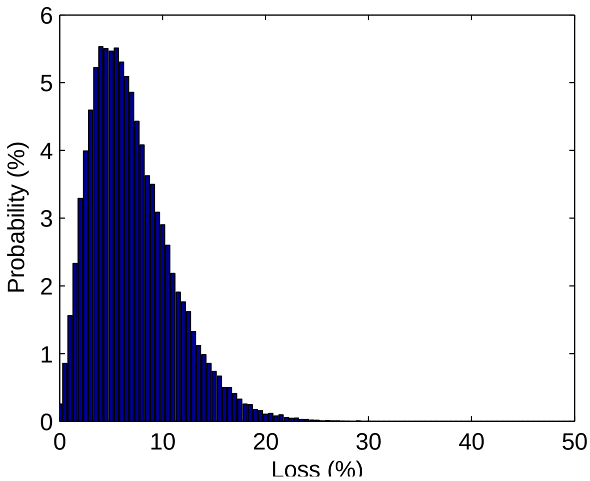

Figure 3.4: Portfolio loss distribution for correlation at 10%, recovery rate 50%, hazard rate 0.03 and 50,000 simulations.

3.2 CDO Spread

Pricing a CDO tranche is very similar to pricing a basket of credit default swaps. First we have to estimate the present value of tranche losses triggered by credit events during the CDO lifetime. Second we calculate the present value of the premium payments weighted by the outstanding capital (original tranche amount minus accumulated losses). The former is called the default leg (DL) and the latter is the premium leg (PL). The fair spread s is defined such that the expected value of both legs is equal. The expectation is taken under the risk-neutral measure.

$$s = \frac{E[DL]}{E[PL]}. \tag{3.15}$$

Let us now describe in more details how to calculate DL and PL for a homogeneous CDO tranche [a,b] with $n = 100$. Consider the k^{th} iteration of a Monte Carlo simulation. The accumulated loss at time t, in percentage, is given by (3.11):

$$\mathcal{L}^k(t) = (1-R)\sum_{i=1}^{100} \mathbb{I}_{\{\tau_i^k < t\}} \ , \tag{3.16}$$

where $\{\tau_1^k, \ldots, \tau_{100}^k\}$ is a sequence of default times obtained as described in section 3.1. We can sort this sequence in increasing order $\{\tau_1'^k, \ldots, \tau_{100}'^k \mid \tau_i'^k < \tau_{i+1}'^k \ \forall\, i\}$. Equation (3.12) gives us the tranche loss, therefore the k^{th} default leg is:

$$DL^k = \sum_{i=1}^{100} e^{-r\tau_i'^k} \left[L_{a,b}(\tau_i'^k) - L_{a,b}(\tau_{i-1}'^k)\right] \mathbb{I}_{\{\tau_i'^k \leq \ T\}}, \tag{3.17}$$

where r is the risk-free interest rate and $L_{a,b}(\tau_0'^k) \equiv 0$.

The premium leg is paid over the outstanding capital in the tranche. If during the lifetime of the CDO the tranche is wiped out, there are no more premium payments. The formula is given by:

$$
\begin{aligned}
PL^k &= N \sum_{j=1}^{w} \delta_j e^{-rt_j} \; min\left\{max\left[b - \mathcal{L}^k(t_j), 0\right], b - a\right\} \\
&= N \sum_{j=1}^{w} \delta_j e^{-rt_j} \left[\left(b - \mathcal{L}^k(t_j)\right)^+ - \left(a - \mathcal{L}^k(t_j)\right)^+\right],
\end{aligned}
\tag{3.18}
$$

where $\{t_1, \ldots, t_w\}$ are the premium payment dates with frequency $\delta_j = t_j - t_{j-1}$ and $(x)^+ \equiv max(x, 0)$.

Taking the average of m values of (3.17) and (3.18) and applying them in (3.15), we obtain the estimate of s for the CDO from Fig. 1.4:

Tranche	s (bps/year)	Std. error (bps/year)
14% - 100%	35.4	0.4
3% - 14%	966	6
0% - 3%	4107	21

Table 3.2: Fair spread of a homogeneous CDO calculated with Monte Carlo with 50,000 simulations, constant hazard rate 0.03, correlation parameter 0.3, recovery rate 0.4 and quarterly premium payments ($\delta = 1/4$).

As expected the equity tranche demands the highest spread because it is the riskiest one. Crude Monte Carlo again gives us a precise estimation with a relative error of the order of 1%. Notice that when calculating the standard error σ_s we have to take into account the covariance between DL and PL:

$$
\sigma_s = \frac{s}{\sqrt{m}} \sqrt{\frac{\sigma_{DL}^2}{(E[DL])^2} + \frac{\sigma_{PL}^2}{(E[PL])^2} - 2\,\frac{cov(DL, PL)}{E[DL]E[PL]}}\,, \tag{3.19}
$$

where σ_{DL}^2 and σ_{PL}^2 are the variance estimators[1].

[1]See proof of eq. (3.19) in appendix A

Figures (3.5) to (3.10) show the spread dependence on correlation and recovery rate. It is natural that they should behave similarly to the expected loss: riskier the tranche, higher the premium demanded by the protection seller. Notice also that the equity and the senior tranches are very sensitive to the recovery rate, therefore indicating that this is an important parameter to estimate when pricing a CDO [14].

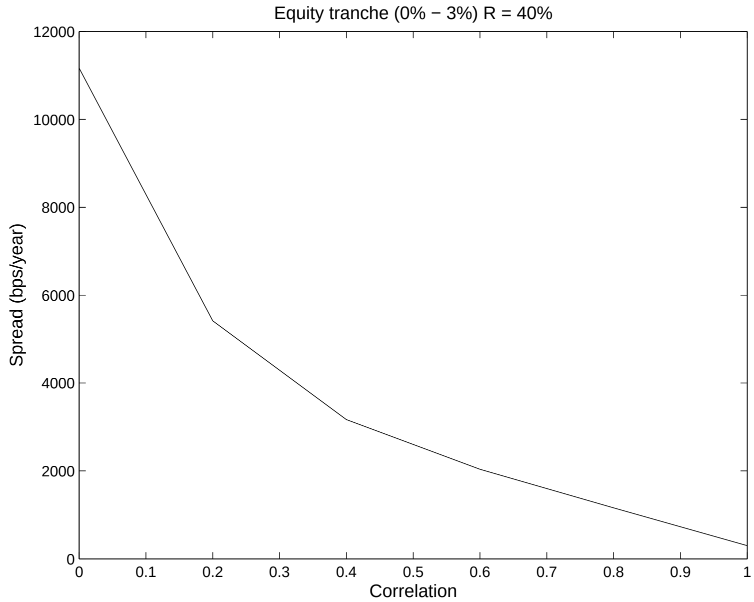

Figure 3.5: Equity spread versus correlation with R = 0.4, h = 0.03 and 20,000 simulations.

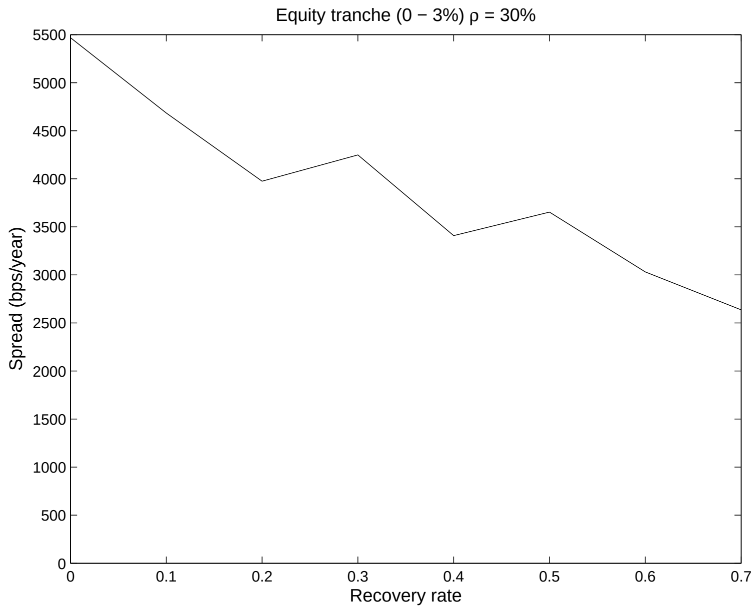

Figure 3.6: Equity spread versus recovery rate with ρ = 0.3, h = 0.03 and 20,000 simulations.

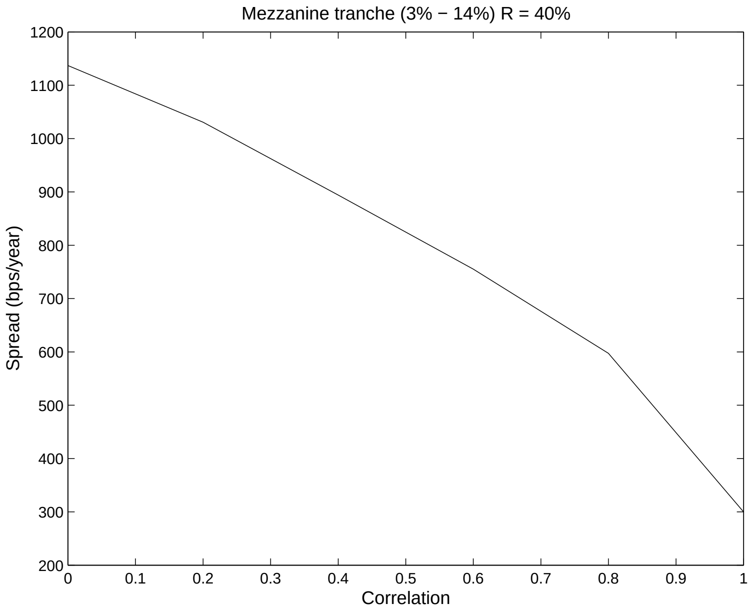

Figure 3.7: Mezzanine spread versus correlation with R = 0.4, h = 0.03 and 20,000 simulations.

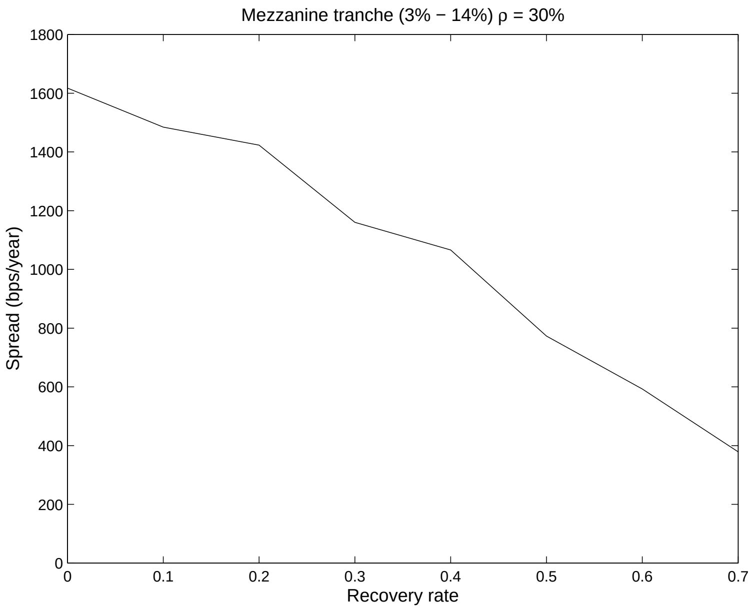

Figure 3.8: Mezzanine spread versus recovery rate with $\rho = 0.3$, h = 0.03 and 20,000 simulations.

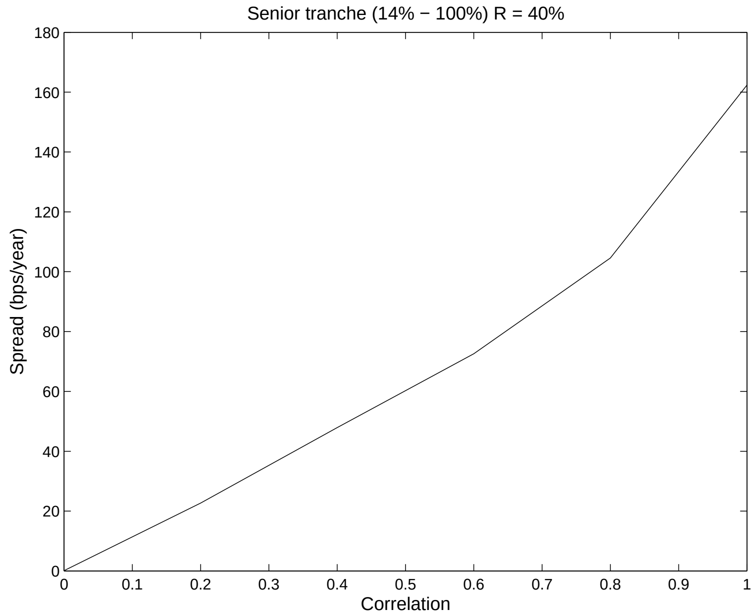

Figure 3.9: Senior spread versus correlation with R = 0.4, h = 0.03 and 20,000 simulations.

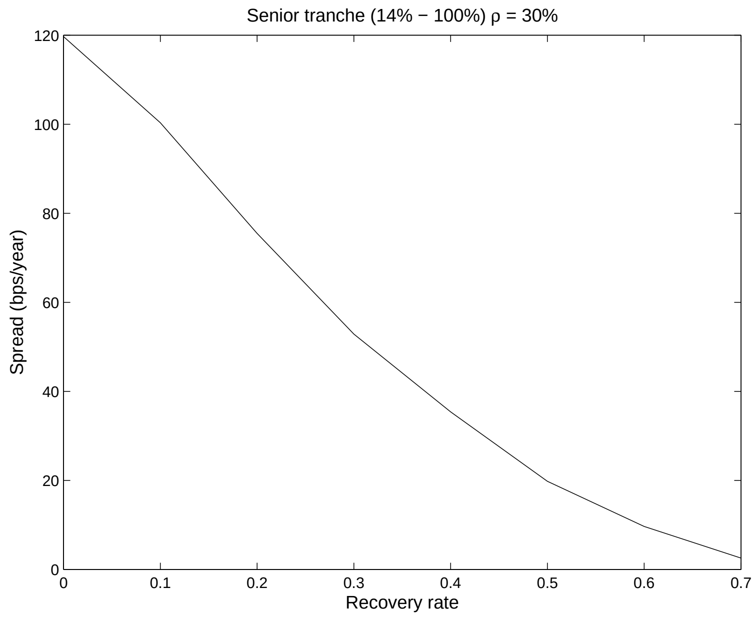

Figure 3.10: Senior spread versus recovery rate with $\rho = 0.3$, h = 0.03 and 20,000 simulations.

Chapter 4

Single Factor Copula Model

Latent factor models are well known in Actuarial Science where they have been used to model loan loss distributions for some time (see references in [1]). Their main property is that the default events, when conditioned on some factor, are independent. This allows the use of semi-analytical methods to estimate the joint default probability and other relevant quantities. Especially in high-dimension problems this could be a good alternative approach to Monte Carlo simulation.

For the particular case of a CDO, a factor approach has been proposed by J.P. Laurent and J. Gregory [1]. The motivation in applying this technique is because CDOs usually contain a large number of obligors (100 or more) where a semi-analytical method could prove to be more time efficient than simulation. Notice however that the factor model is not purely analytical because a numerical integration is still necessary.

In this report we will study the application of a one factor model with a Gaussian copula to a homogeneous CDO. We start by assuming a simplified version of a firm's value model [5].

Assumption 4.0.1 *The default of each obligor is triggered by the change of value of its assets. This value is denoted by $V_i(t)$ and we assume that it is normalized and it has a standard normal distribution $\mathcal{N}(0,1)$. The assets of each firm is correlated with each other through a correlation matrix Σ. Each obligor defaults when $V_i \leq K_i$, where K_i is a stochastic barrier.*

Even with this simple model there are still $N(N-1)/2$ parameters to be estimated in Σ for a portfolio with N instruments. This can be a daunting task as in the case with $N = 100$, where $N(N-1)/2 = 4950$. Therefore we need to make more assumptions. A common one is to assume that all the firm values V_i depend on M common factors. In this case the number of free parameters is $M \times N$. Let us assume a one factor model ($M = 1$) as proposed in [1]:

Assumption 4.0.2 *The firm values are driven by a common factor V and a noise term ϵ_i.*

$$V_i(t) = a_i V + \sqrt{1 - a_i^2}\epsilon_i, \tag{4.1}$$

where V and ϵ_i are independent, standard normal random variables.

From Eq.(4.1) we get:

$$Cov(V_i, V_j) = a_i a_j, \tag{4.2}$$

$$Var(V_i) = 1. \tag{4.3}$$

Therefore in this model there are only N free parameters in Σ. Later on we will simplify even further by assuming that $a_i = a_j = a,\ \forall i \neq j$, just like we did in section 3.1. The factor V can be understood as an underlying economic variable, like the business cycle or the interest rate, that affect different industries at the same time. The noise ϵ_i is related to firm specific factors as management, balance sheet, etc. The important thing is that, conditioned on V, the firm's values and default distribution are independent.

Since a default is triggered by V_i crossing the barrier K_i, we can say that the distribution of default time is:

$$F_i(t) = P(\tau_i \leq t) = P(V_i \leq K_i), \tag{4.4}$$

$$K_i = \Phi^{-1}[F_i(t)], \tag{4.5}$$

where we used the assumption that $V_i \sim \mathcal{N}(0, 1)$.

The conditional probability of default is:

$$p_i(t|V) = P(V_i \leq K_i|V) = \Phi\left[\frac{\Phi^{-1}[F_i(t)] - a_i V}{\sqrt{1 - a_i^2}}\right], \tag{4.6}$$

where we have used (4.1), (4.5) and the fact that $\epsilon_i \sim \mathcal{N}(0, 1)$. Using the assumption that the defaults are independent when conditioned on V, we obtain the joint probability of default:

$$F(t_1, \ldots, t_n) = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \Phi\left[\frac{\Phi^{-1}[F_i(t)] - a_i v}{\sqrt{1 - a_i^2}}\right] g(v)dv, \tag{4.7}$$

$$S(t_1, \ldots, t_n) = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \Phi\left[\frac{a_i v - \Phi^{-1}[F_i(t)]}{\sqrt{1 - a_i^2}}\right] g(v)dv, \tag{4.8}$$

where $g(v) = e^{-v^2/2}/\sqrt{2\pi}$ is the density function of V. Equations (4.7) and (4.8) can be integrated using numerical methods and they allow us to price any kind of credit portfolio.

4.1 Basket Default Swap

Because of their pricing similarities, we will first apply the one factor model to a basket default swap before pricing a CDO. We consider a k^{th}-to-default swap that only pays when the k^{th} default happens in a portfolio with $n \geq k$ obligors. In order to price this instrument, we need to know the risk-neutral probability of k defaults by time t. Equation (4.8) can be written as:

$$S(t_1, \ldots, t_n) = \int_{-\infty}^{\infty} \prod_{i=1}^{n} q_i(t|v) g(v) dv, \tag{4.9}$$

$$q_i(t|v) = \frac{a_i v - \Phi^{-1}[F_i(t)]}{\sqrt{1 - a_i^2}}, \tag{4.10}$$

where $q_i(t|V)$ is the conditional survival function of instrument i. The number of defaults at time t is given by the counting process (2.7) and in a one factor model the probability of k defaults can be written as:

$$P[N(t) = k] = \int P[N(t) = k|v] g(v) dv. \tag{4.11}$$

Therefore we have to find a formula for $P[N(t) = k|V],\ k \in \{0, \ldots, n\}$. Let us start with the conditional probability of no-default:

$$P[N(t) = 0|V] = \prod_{i=1}^{n} q_i(t|V). \tag{4.12}$$

The conditional probability of one default is:

$$P[N(t) = 1|V] = \sum_{i=1}^{n} \left[(1 - q_i) \prod_{\substack{j=1 \\ j \neq i}}^{n} q_j \right], \tag{4.13}$$

$$P[N(t) = 1|V] = \sum_{i=1}^{n} \left[(1 - q_i) \prod_{j=1}^{n} \frac{q_j}{q_i} \right]. \tag{4.14}$$

Using (4.12) in (4.14):

$$P[N(t) = 1|V] = P[N(t) = 0|V] \sum_{i=1}^{n} w_i(t|V), \tag{4.15}$$

$$w_i(t|V) = \frac{1 - q_i(t|V)}{q_i(t|V)}. \tag{4.16}$$

Repeating a similar procedure for $k = \{2, 3, \ldots\}$, we can quickly verify that:

$$P[N(t) = k|V] = P[N(t) = 0|V] \sum w_{p(1)} \ldots w_{p(k)} \ , \tag{4.17}$$

where the sum is taken over all $n!/(n-k)!k!$ different combinations of $\{1, \ldots, n\}$. Reference [3] gives an algorithm to calculate all the elements of this sum. Calling $A_k = \sum w_{p(1)} \ldots w_{p(k)}$ and $B_j = \sum_{i=1}^{n} (w_i)^j$, they prove the following recurrence relation:

$$A_1 = B_1, \tag{4.18}$$

$$2A_2 = B_1 A_1 - B_2, \tag{4.19}$$

$$3A_3 = B_1 A_2 - B_2 A_1 + B_3, \tag{4.20}$$

$$kA_k = B_1 A_{k-1} - B_2 A_{k-2} + B_3 A_{k-3} - \ldots + (-1)^k B_{k-1} A_1 + (-1)^{k+1} B_k. \tag{4.21}$$

The algorithm (4.18) - (4.21) can be easily implemented to generate A_k for all $k \in \{0, \ldots, n\}$. Knowing how to calculate (4.11) is enough to price any basket default swap, because the k^{th} survival function can be written as:

$$S^k(t) = P(\tau^k > t), \tag{4.22}$$

$$S^k(t) = P[N(t) < k], \tag{4.23}$$

$$S^k(t) = \sum_{j=0}^{k-1} P[N(t) = j], \tag{4.24}$$

where τ^k is the k^{th} default time. For the particular case of a homogeneous basket with flat correlation, equation (4.17) simplifies considerably and we do not need to use the recurrence relation (4.21). The results are given in appendix B.

When pricing a basket we assume that interest rates and recovery rates are deterministic and that they are independent of default times. Similarly to what we did in chapter 3, we have to calculate the premium leg (PL) and the default leg

(DL) for a k^{th}-to-default swap. The expected value of the premium leg is given by the expected value of the discounted premium cashflows:

$$E[PL_k] = E\left[s_k \sum_{i=1}^{I} \delta_i \beta(t_i) \mathbb{I}_{\{\tau^k > t_i\}}\right], \tag{4.25}$$

where s_k is the k^{th}-to-default premium, t_i are the premium payment dates, $\delta_i = t_i - t_{i-1}$ are the payment frequencies and $\beta(t)$ is the discount factor. Using (4.24) in (4.25) we get:

$$E[PL_k] = s_k \sum_{i=1}^{I} \left[\delta_i \beta(t_i) \sum_{j=0}^{k-1} P[N(t_i) = j]\right]. \tag{4.26}$$

In order to calculate the default leg, we will assume a homogeneous basket where every obligor has the same notional value of 1 and recovery rate R. This simplifies the calculation, since now we only need to know (4.24) to price it. The expected value of the default leg is given by the expected value of any default before maturity:

$$E[DL_k] = E[(1-R)\beta(\tau^k)\mathbb{I}_{\{\tau^k \leq T\}}] = -(1-R)\int_0^T \beta(t)\ dS^k(t), \tag{4.27}$$

where T is the maturity date. Integrating (4.27) by parts:

$$E[DL_k] = (1-R)\left[1 - \beta(T)S^k(T) + \int_0^T S^k(t)\ d\beta(t)\right]. \tag{4.28}$$

Using our assumption of deterministic interest rates, under the risk-neutral measure we have:

$$E[DL_k] = (1-R)\left[1 - e^{-rT}S^k(T) - r\int_0^T S^k(t)e^{-rt}dt\right]. \tag{4.29}$$

The premium for the basket is given by the ratio of (4.29) and (4.26):

$$s_k = \frac{E[DL_k]}{E[PL_k]}. \tag{4.30}$$

Table 4.1 shows the spread (4.30) in bps/year for the k^{th} default from a basket with 10 obligors. We use a single parameter $a = \sqrt{0.3}$ for the correlation matrix and the recovery rate is constant at 40%. Notice that from (4.2) we have $a = \sqrt{\rho}$, where ρ is the correlation parameter (3.7) used in the Monte Carlo simulation. Table 4.2 shows the spread in bps/year for different correlation factors, with a fixed hazard rate 0.03.

k	**h = 0.01**	**h=0.03**
1	445	1194
2	140	519
3	53	266
4	21	141
5	8	73
6	3	36
7	1	16
8	0.3	6
9	0.1	2
10	0	0.4

Table 4.1: Spreads for a k^{th} to default basket swap on a homogeneous portfolio with 10 instruments, fixed correlation factor $a = \sqrt{0.3}$ and recovery rate 40%.

These tables agree with what we would expect. In table 4.1 a higher hazard rate means a larger spread for all swaps, because the chance of default increases. Table 4.2 shows that increasing the correlation among the instruments in the portfolio causes early default swaps to get cheaper and late ones to get more expensive. This is similar to the behavior observed in the CDOs in Figs. 3.5 and 3.9. This is

k	$\mathbf{a = 0}$	$\mathbf{a = \sqrt{0.3}}$	$\mathbf{a = \sqrt{0.6}}$
1	1880	1194	755
2	596	519	421
3	184	266	277
4	45	141	192
5	8	73	135
6	1	36	93
7	0	16	63
8	0	6	40
9	0	2	22
10	0	0.4	9

Table 4.2: Spreads for a k^{th} to default basket swap on a homogeneous portfolio with 10 instruments with fixed hazard rate 0.03 and recovery rate 40%.

an evidence that their pricing should be similar as we will see in the next section.

4.2 CDO spread

For a homogeneous portfolio with zero recovery rate, a CDO tranche is equivalent to a basket default swap. For the general case of recovery rate different than zero, the stochastic process that we have to observe is the total portfolio loss $L(t)$ (3.4). The premium leg for a tranche $[A, B]$ is:

$$PL = \sum_{i=1}^{I} \delta_i \beta(t_i) \left\{ (B - A)\mathbb{I}_{\{L(t_i) \in [0,A]\}} + [B - L(t_i)]\mathbb{I}_{\{L(t_i) \in [A,B]\}} \right\} . \tag{4.31}$$

Taking the expected value of (4.31) we have:

$$E[PL] = \sum_{i=1}^{I} \delta_i \beta(t_i) \left\{ (B-A) P[L(t_i) < A] + \int_A^B (B-x)\, dF_L(x) \right\}, \tag{4.32}$$

where F_L is the distribution of $L(t)$.

If we consider a homogeneous CDO with each notional being 1, the set of all possible values for $L(t)$ is $\{0, (1-R), \ldots, n(1-R)\}$. Therefore we can make an association between $P[L(t) = C]$ and $P[N(t) = k]$ from (4.11):

$$P[L(t) = C] = P\left(N(t) = \left\lfloor \frac{C}{1-R} \right\rfloor + 1 \right) , \tag{4.33}$$

where $\lfloor x \rfloor = max\{m \in \mathbb{Z} | m \leq x\}$ is the floor function.

Using (4.33) we can re-write (4.32) as:

$$E[PL] = \sum_{i=1}^{I} \delta_i \beta(t_i) \left\{ (B-A) \sum_{k=0}^{\lfloor \frac{A}{1-R} \rfloor} P[N(t_i) = k] + \sum_{k=\lfloor \frac{A}{1-R} \rfloor + 1}^{m_1} [B - k(1-R)] P[N(t_i) = k] \right\}, \tag{4.34}$$

where $m_1 = \min(\lfloor \frac{B}{1-R} \rfloor,\ n)$.

Now we have to calculate the default leg. The loss accumulated in a tranche is given by:

$$M(t) = [L(t) - A] \mathbb{I}_{\{L(t) \in [A,B]\}} + (B-A) \mathbb{I}_{\{L(t) > B\}}. \tag{4.35}$$

Since $M(t)$ is an increasing jump process, we can define a Riemann-Stieltjes integral to represent the default leg [1]:

$$DL = \int_0^T \beta(t)\, dM(t). \tag{4.36}$$

Integrating (4.36) by parts and taking into account that $\beta(t) = e^{-rt}$ we have:

$$E[DL] = e^{-rT}\ E[M(T)] + r \int_0^T e^{-rt}\ E[M(t)]\ dt. \tag{4.37}$$

Tranche	MC	Factor model
14% - 100%	35.4 ± 0.4	35.1
3% - 14%	966 ± 6	969
0% - 3%	4107 ± 21	4092

Table 4.3: Comparison of spreads calculated with 50,000 Monte Carlo simulations and a single factor model. CDO has 100 obligors with h = 0.03 each, $a = \sqrt{0.3}$ and $R = 40\%$.

The expected value of (4.35) is:

$$E[M(t)] = (B - A)P[L(t) > B] + \int_A^B (x - A)\; dF_L(x). \tag{4.38}$$

Applying (4.33) in (4.38):

$$E[M(t)] = (B-A)\mathbb{I}_{\{B<N_T\}} \sum_{k=m_2}^{n} P[N(t) = k] + \sum_{k=\lfloor \frac{A}{1-R} \rfloor + 1}^{m_1} [k(1-R) - A]\, P[N(t) = k], \tag{4.39}$$

where $m_2 = \min(\lfloor \frac{B}{1-R} \rfloor + 1,\ n)$.

Using (4.11), (4.34), (4.37), (4.39) and appendix B we can calculate the tranche spread for a homogeneous CDO. Notice that for a super-senior tranche ($B = N_T$) the first term on the right hand side of Eq. (4.39) is zero. Table 4.3 compares the spreads from table 3.2 with the ones obtained with a single factor model. The values obtained with the factor model agree with the Monte Carlo simulation within one standard deviation.

Chapter 5

Conclusion

In this report we have priced a homogeneous CDO using Monte Carlo simulation and a semi-analytical method. Both prices agreed within one standard deviation, indicating that the factor model is a valid alternative when pricing a CDO.

The main advantage of the factor model is that it can be much faster than Monte Carlo simulation. The calculations in this report were done using Matlab. For a homogeneous portofolio with 100 names, the Monte Carlo program with 50,000 simulations took 27 seconds to calculate the spread of a CDO tranche. The same portfolio took 4 seconds with the factor model. This is a significant gain of a factor of 7. We have noticed a trade off in time vs. accuracy in the analytical method: greater accuracy in the numerical integration means longer calculation time. Greater accuracy is needed when using large correlation values ($\rho > 0.8$).

An important point that deserves attention is the calculation of $P[N(t) = k|V]$ (4.17). For a homogeneous basket with flat correlation it is trivial (see appendix B). In a general case however, when $q_i(t|V)$ goes to zero in (4.12), $P[N(t) = 0|V]$ approaches zero also and $w_i(t|V)$ in (4.16) goes to infinity. It is possible that in this situation a floating point error could cause the product of $P[N(t) = 0|V]$ and $\sum w_{p(1)} \ldots w_{p(k)}$ in (4.17) to be very large, eventhough it should be upper bounded by one by the definition of probability. This can be a source of error when estimating the conditional default probability using the recursive algorithm (4.18) - (4.21) and it is usually obvious what the correct value should be (in most cases it is zero).

A last observation to be made is that the gain in speed has been checked only for the special case of a homogeneous CDO with flat correlation. In the non-homogeneous situation things get more complicated and the gain will probably not be as good. However the factor model is still a valid way to double check results obtained with Monte Carlo and, due to its semi-analytical nature, it has always the potential to speed up the calculation dramatically if it is programmed in an efficient way.

Appendix A

Proof of the Error Propagation Equation

In this appendix we derive equation (3.19) used to estimate the Monte Carlo error for the CDO spread. It is based on the error propagation equation commonly used in experimental Physics [15].

Suppose a derived quantity z is a function of n variables $\{x_i\}$.

$$z = f(x_1, \ldots, x_n). \tag{A.1}$$

We want to estimate the standard deviation σ_z as a function of $\sigma_i \equiv \sigma_{x_i}$. By definition the variance σ_z^2 of an N sample is given by:

$$\sigma_z^2 = \frac{1}{N}\sum_{j=1}^{N}(z_j - \overline{z})^2 . \tag{A.2}$$

From Eq. (A.1) we have:

$$z_j = f(x_{1j}, \ldots, x_{nj}), \tag{A.3}$$

$$\overline{z} \cong f(\overline{x}_1, \ldots, \overline{x}_n), \tag{A.4}$$

where (A.4) is an approximation. Using Taylor expansion in (A.3) around the point $(\overline{x}_1, \dots, \overline{x}_n)$ and assuming (A.4) we get:

$$z_j - \overline{z} = \sum_{i=1}^{n} (x_{ij} - \overline{x}_i) \left.\frac{\partial z}{\partial x_i}\right|_{\overline{x}_i} + O(2). \tag{A.5}$$

Applying (A.5) in (A.2) and ignoring second order terms we have:

$$\begin{aligned}
\sigma_z^2 &= \sum_{i=1}^{n} \frac{1}{N} \sum_{j=1}^{N} (x_{ij} - \overline{x}_i)^2 \left(\left.\frac{\partial z}{\partial x_i}\right|_{\overline{x}_i} \right)^2 \\
&+ 2 \sum_{\substack{i,k=1 \\ i<k}}^{n} \frac{1}{N} \sum_{j=1}^{N} (x_{ij} - \overline{x}_i)(x_{kj} - \overline{x}_k) \left(\left.\frac{\partial z}{\partial x_i}\right|_{\overline{x}_i} \right) \left(\left.\frac{\partial z}{\partial x_k}\right|_{\overline{x}_k} \right), \\
\sigma_z^2 &= \sum_{i=1}^{n} \sigma_i^2 \left(\left.\frac{\partial z}{\partial x_i}\right|_{\overline{x}_i} \right)^2 + 2 \sum_{\substack{i,k=1 \\ i<k}}^{n} cov(x_i, x_k) \left(\left.\frac{\partial z}{\partial x_i}\right|_{\overline{x}_i} \right) \left(\left.\frac{\partial z}{\partial x_k}\right|_{\overline{x}_k} \right). \qquad (A.6)
\end{aligned}$$

Expression (A.6) is known as the error propagation equation [15]. For the particular case $z = x/y$, (A.6) gives:

$$\frac{\sigma_z^2}{\overline{z}^2} = \frac{\sigma_x^2}{\overline{x}^2} + \frac{\sigma_y^2}{\overline{y}^2} - 2\frac{cov(x, y)}{\overline{xy}}. \tag{A.7}$$

Equation (3.19) is the application of (A.7) to the credit spread (3.15).

Appendix B

Homogeneous Basket with Flat Correlation

For the particular case of a homogeneous basket with flat correlation, the conditional default probability (4.17) becomes the binomial expansion. This can be shown by noticing that in this situation the conditional survival function defined in (4.10) is the same for all names ($q \equiv q_i(t|V)\ \forall\ i$), giving us:

$$P[N(t) = 0|V] = q^n, \tag{B.1}$$

$$w \equiv w_i(t|V) = \frac{1-q}{q}, \tag{B.2}$$

$$A_k = \sum w_{p(1)} \ldots w_{p(k)} = \frac{n!}{(n-k)!\ k!}\, w^k \equiv \begin{pmatrix} n \\ k \end{pmatrix} w^k. \tag{B.3}$$

Using (B.1) - (B.3) in (4.17) we obtain the conditional default probability for the basket:

$$P[N(t) = k|V] = \begin{pmatrix} n \\ k \end{pmatrix} q^{n-k}(1-q)^k, \tag{B.4}$$

where $k \in \{0, \ldots, n\}$.

For the extreme cases $q = 0$ and $q = 1$ we have:

- $\mathbf{q = 0}$

$$P[N(t) = k|V] = 0, \; k \neq n \tag{B.5}$$

$$P[N(t) = n|V] = 1. \tag{B.6}$$

- $\mathbf{q = 1}$

$$P[N(t) = k|V] = 0, \; k \neq 0 \tag{B.7}$$

$$P[N(t) = 0|V] = 1. \tag{B.8}$$

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